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PROCEEDINGS

OF THE

AMERICAN ACADEMY

OF

ARTS AND SCIENCES.

NEW SERIES.
Vol. V.

WHOLE SERIES.
Vol. XIII.

FROM MAY, 1877, TO MAY, 1878.

SELECTED FROM THE RECORDS.

BOSTON:
PRESS OF JOHN WILSON AND SON.
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PROCEEDINGS

OF THE

AMERICAN ACADEMY

OF

ARTS AND SCIENCES.

VOL. XIII.

PAPERS READ BEFORE THE ACADEMY.

I.

REVISION OF THE ATOMIC WEIGHT OF ANTIMONY.

BY JOSIAH P. COOKE, JR.,
Erving Professor of Chemistry and Mineralogy in Harvard College.

THE first determination of what we now call the atomic weight of antimony was made by Berzelius, and the result published in 1818.* His method consisted in oxidizing the metal with nitric acid and igniting the residue. This product he called antimonious acid; and, after a careful study of the chemical relations of the several oxides and of the native sulphide of antimony, he assigned to antimonious acid the symbol $Sb\,O_4$, that of the basic oxide being $Sb\,O_3$, and of the native sulphide $Sb\,S_3$. In some earlier experiments,† Berzelius had not obtained constant results with the same process, 100 parts of "pure antimony" having yielded in four experiments 125.8, 126, 127.5, and 127.8 parts of antimonious acid respectively; but in his later paper he attributes the discordance to the circumstance that he had previously conducted the process in glass flasks, "which could not bear a sufficiently strong heat to change all the yellow residue into white oxide." By the later experiments, in which platinum vessels were used, he found that 100 parts of antimony yield 124.8 parts of antimonious acid. But although he distinctly states "that when pure antimony is oxidized in a flask with nitric acid, and the mass evaporated to dryness in a platinum crucible is heated until it becomes perfectly white, the same results are always obtained," nevertheless, in the paper cited, he does not give the several determinations on which his conclusion was based, or any details by which we can now judge of the purity of the metal used. He simply gives the single result just cited, from which we readily deduce as the atomic weight of antimony 129; and with this value he appears to have been contented; for we find in his masterly review of the atomic weights of the chemical elements, published in 1826,* the same value given, without further comment, and it remained unquestioned for more than thirty-eight years.

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* Schweigger Jour. für Chem. und Phys., xxii. 70, 1818.

† Ibid., vi. 149, 1812.

Not until 1856 does any further investigation of the subject appear to have been made. In February of that year, R. Schneider, of Berlin, published a preliminary notice;† and later, in May, the full details‡ of a new determination of the atomic weight of antimony, made by a method wholly different from that of Berzelius. This method consisted in reducing a pure native sulphide of antimony by hydrogen gas. The material selected was the antimony glance of Arnberg, which is distinguished for its high degree of purity, and by appropriate tests the mineral was shown to be free from arsenic and the metals which usually accompany antimony. The only impurity that could be discovered was a small amount of quartz, which is associated with the mineral as gangue; but this was of no importance, since the amount could be determined in every experiment with almost absolute precision. The same was almost equally true of two well-known defects in the process, which, if not allowed for, would become sources of error; namely, the escape in vapor, or, mechanically, of a small amount of sulphide of antimony during the reduction, and the circumstance that a further small amount of sulphide escapes reduction by becoming enclosed in the regulus. The first of these effects would diminish, and the second increase, the weight of the regulus; and the same effects would also respectively diminish or increase the atomic weight of antimony calculated from the observations. The two sources of error evidently tend to balance each other; but, on the other hand, the effect of the last is several times greater than that of the first. The total effect, however, is but small; and, had these known sources of error been wholly overlooked by Schneider, the effect would only have been to raise the mean result about 0.26. So far, however, from neglecting these very small errors, Schneider in every case accurately collected and determined both the sulphide of antimony volatilized and the sulphur retained by the regulus, and with these amounts, never exceeding a few milligrammes, he corrected his observed data. In the paper referred to, Schneider gives the detail of eight separate determinations, the mean of which gives for the atomic weight of antimony the value 120.30, and this differs from the extremes on either side by only 0.23. In this determination, the atomic weight of antimony is referred to that of sulphur, which is taken as 32.*

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* Poggendorff's Annalen, viii. 23, 1826.

† Ibid., xcvii. 483, February, 1856.

‡ Ibid., xcviii., 293, May, 1856.

In the next number of Poggendorff's "Annalen" to that in which the paper of Schneider appeared, Heinrich Rose published the result of a previous analysis of $Sb\,Cl_3$, made under his direction by Herr Weber, which gives for the atomic weight of antimony the value 120.7 when chlorine is 35.5; and, although we have only the evidence of a single analysis, yet this one was thought by Rose to be of especial value as confirming the subsequent determination of Schneider. This confirmation he regarded as the more satisfactory, since by Weber's analysis the atomic weight of antimony is referred to chlorine, and not as before to sulphur or oxygen. The process was this: A weighed amount of $Sb\,Cl_3$, purified by repeated fractional distillation, was dissolved in an aqueous solution of tartaric acid. From this solution the metal was precipitated by H_2S, and in the filtrate the chlorine was determined by precipitation with $AgNO_3$ in the usual way.

Next in order in this series of investigations comes the admirable work of Mr. W. P. Dexter. He adopted the same process as Berzelius, but conducted it with all the refinement which ingenuity, guided by the increased chemical knowledge of the time, could devise. Especial care was taken to secure pure metallic antimony; and a comparison of the results obtained with different specimens — purified by the processes he describes — shows conclusively that, even if the metal was not absolutely pure, the error resulting from this cause was wholly inappreciable. The agreement between the results of the several experiments is wonderfully close, but for the details we must refer to the original paper.† It is only important to state here that Dexter obtained for the atomic weight of antimony, as the mean of ten determinations, the value 122.34, and that the extremes on either side were

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* Poggendorff's Annalen, xcviii. 455, June, 1856.
† Ibid., c. 563, April, 1857.

122.24 and 122.48. The difference between these extremes corresponds to only 1 milligramme in 3 grammes of Sb_2O_4, which was about the average amount weighed in these experiments.

In June, 1861, F. Kessler, of Dantzic, published in Poggendorff's "Annalen" a paper * devoted to a re-examination of the atomic weights of chromium, arsenic, and antimony, based on a method of volumetric analysis which he had previously described in the same journal.† Indeed, in his earlier paper he had already quite fully elaborated the subject, and thus anticipated both Schneider and Dexter in correcting the old number of Berzelius. The method of Kessler was based on the reciprocal relations of potassic dichromate and ferrous chloride; and, as is well known, standard solutions of these salts are especially well adapted to volumetric determinations, by the circumstance that the neutral point can be so accurately fixed by the reaction with potassic ferricyanide. For the details of the method, which were quite numerous and complicated, we must refer to the original paper. It is sufficient for the present purpose to say that the atomic weight of antimony was thus indirectly referred to the molecular weight of potassic chlorate, which was taken as 122.57, according to the determinations of Pelouze and of Marignac. Kessler experimented with antimony, with antimonious oxide, and antimonious chloride, and obtained results varying between 121.67 and 122.61. We translate his own commentary: "Although the results of these experiments, taken as a whole, agree very nearly with those obtained by Dexter, I must nevertheless confess that I am not wholly convinced that our number, which is two hydrogen units higher than that found by Schneider, is much nearer the truth than his. Every one who has occupied himself with the analysis of different antimony compounds must be able to indorse what Berzelius wrote in 1812: 'I have never worked with a material with which it was so extremely difficult to obtain constant results.' . . . Nevertheless, I believe I have been able to show that in the analysis of antimony compounds the volumetric method is capable of affording a very sharp control over results obtained in other ways."

Before the Académie des Sciences in Paris, in 1857 and 1858,‡ Dumas read the results of his celebrated revision of the atomic weights of the chemical elements, during which he redetermined the atomic weight of antimony.§ With this, as with most of the other elements

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* Poggendorff's Annalen, cxiii. 134, June, 1861.

† Ibid., xcv. 204, June, 1855.

‡ The 9th of November, 1857, and the 27th of December, 1858.

§ Annales de Chimie et de Physique, 3me Séries, LV. 175, February, 1859.

whose atomic weights he redetermined, his process consisted in analyzing the chloride ($SbCl_3$) and weighing the chlorine as argentic chloride. The antimonious chloride was prepared by the action of dry chlorine on pure antimony, — "de l'antimoine très-pur et du chlor sec," — and purified by repeated distillations over the metal in fine powder. A weighed amount of this chloride was added to a solution of tartaric acid in water, and the chlorine directly precipitated in the usual way. Dumas adds: "Je n'ai entrevu ni apparence de cause d'erreur particulière, ni irrégularité dans les résultats." He gives four determinations as follows: —

1. 1.876 of $SbCl_3$ correspond to 2.660 Ag; hence Sb = 122.00
2. 4.336 " " 6.148 " " = 122.00
3. 5.065 " " 7.175 " " = 122.20
4. 3.475 " " 4.930 " " = 121.94

The first three were from the same, but the last from a different preparation. From these results, Dumas deduces as the most probable value of the atomic weight of antimony the whole number 122, and this differs from the extreme value obtained by only 0.2. It will be noticed that the method of Dumas is essentially the same as that of Rose, only the former did not separate the antimony from the solution before precipitating with argentic nitrate. This, however, may be an important difference; for, as we shall show hereafter, an excess of argentic nitrate, added to a solution containing antimonious chloride and tartaric acid, above the amount necessary to precipitate the chlorine, determines the formation of argento-antimonious tartrate, which slowly crystallizes out from the solution on standing, and is almost wholly insoluble in cold water. The occlusion of this material by the precipitated argentic chloride on the one hand, and on the other hand the slight solubility which we also noticed of this precipitate in the resulting menstruum, gave us a very different impression of the accuracy of the process from that obtained by Dumas. Rose, however, does not mention these possible sources of error. It seems probable that Weber, under his direction, aimed to estimate the antimony as well as the chlorine of the compound analyzed, and for this reason precipitated the antimony first. Moreover, the chief of the causes of error we have noticed would tend to lower the observed atomic weight of antimony; and our experience would lead us to question whether the previous precipitation of the antimony would not occasion more error than it prevents.

The only other determination of the atomic weight of antimony which has come to our knowledge was made incidentally by B. Unger* while studying the chemical relations of Schlippe's salt, $Na_3S_4Sb\text{-}9\,H_2O$. An analysis of this salt gave $Sb = 119.76$, when $S = 32$ and $Na = 23$; but this value rests on a single experimental determination, by a method from which great accuracy could not be expected.

As is well known, the value 122 obtained by Dumas is the one which is almost universally accepted as the atomic weight of antimony, and this not only in consequence of the deservedly great authority of this distinguished experimenter, but also because his result so nearly agrees with the previous determination of Dexter. Nevertheless, a careful examination of the work of Schneider, referred to above, will convince every chemist that it is impossible to refer the difference between 120.3, the number he obtained, and 122, to any experimental errors. This difference is nearly one and a half per cent of the whole value; although it is evident that the probable experimental error does not, in either case, amount to one tenth of one per cent. Moreover, it will be found that this investigation of Schneider is a model of its kind. All the details of the experimental work are given, and it is evident that every precaution was taken which the circumstances required. Furthermore, the method of Schneider has a very great advantage over all the processes by which the atomic weight of antimony has been determined, in that he was able to obtain satisfactory evidence, not only of the purity, but also of the definite atomic composition of the material he analyzed. In the determination of an atomic weight, it is not only essential that the substance selected for analysis should be pure, in the sense of containing no adventitious elements: it is even more important that neither of the constituents whose atomic ratio is to be determined should be present in excess of the normal proportion. But, while experimenters have been most careful to establish the purity of their material in the first sense, they have seldom given a thought to the possibility of such an impurity as the preponderance of one or the other of the proper constituents of the compound practically constitutes. Theoretical considerations, based on the law of definite proportions, might lead a chemist to believe that impurity in the last sense was impossible in a true chemical compound. But how are we to know that a given material is of the true type? It is certain that the ordinary physical tests of purity, such as the melting point and the boiling point, are insufficient; and the same is equally true of the perfection of crystalline form, that character on which of all others the chemist most frequently relies. As early as 1860, we ourselves showed that constancy of crystalline form was, under certain conditions, compatible with quite wide variations of compositions; * and our conclusions have been most fully confirmed by the results of investigations since made, which reveal the great complexity of crystalline structures. In the compounds † of zinc and antimony which we investigated, the variation was large, because the affinity was weak; but we gave abundant evidence that similar variations, although of less extent, are not uncommon in more stable compounds, and from our subsequent experience we feel confident that in proportion as analytical processes become more accurate these variations will become more apparent. At present, they are to a great extent concealed, because the possible errors of our analytical processes are so great; and we very properly refer, at the outset, any seeming variations from exact atomic proportions to errors of this class. But, in the more accurate methods on which we should alone rely for fixing atomic weights, such variations, when they exist, become of great importance. For call by whatever name you please that small portion of either constituent of a compound which may be present in a crystal over and above the atomic ratio, — regard it merely as so much "dirt" entangled by the crystalline structure, —

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* Archiv. der Pharmacie, 1871, Band 197, S. 194.

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* Crystalline Form not Necessarily an Indication of Definite Chemical Composition. Philosophical Magazine, June, 1860; also Poggendorff's Annalen, Band cxii. 90; also Memoirs of American Academy of Arts and Sciences, New Series, vol. v. p. 337.

† Rammelsberg has expressed the opinion — Pogg. Ann., cxx. 61, 1863 — that all crystalline alloys are isomorphous mixtures of the constituent metals, and includes under this category the two compounds of zinc and antimony here referred to; but, had this accurate observer repeated our experiments and measurements, he would have seen that the whole order of the phenomena we studied is inconsistent with such an assumption. Although the two metals may be alloyed in all proportions, yet we give in our paper the strongest evidence that the union is attended with definite chemical action, which has two maxima at the points represented by the symbols Sb_2Zn_2 and Sb_2Zn_3; and although the measurements of the two sets of crystals may be referred approximately to the same fundamental form, as is frequently the case in the simpler systems, especially if we admit such ratios as 4:5, yet the two types of crystals differ completely in their habit, and the very remarkable circumstances of their formation prove that they are essentially distinct. For a full statement of these circumstances we must refer to our original paper above cited. In the very partial abstract from this paper — Pogg. Ann., xcvi. 584 — to which Rammelsberg alone refers, none of these important facts were given; and, moreover, Rammelsberg does not seem to have noticed that our paper published in 1855 preceded Schneider's determination of the atomic weight of antimony.

yet so long as it is there, and your analytical processes do not distinguish it from that portion of the same element which is chemically combined, it is obvious that impurity of this kind will vitiate your result to a far greater extent than an equal amount of wholly foreign admixture; and of what use can it be to refine processes or multiply and discuss observations, if so large a door is left open to constant errors? Moreover, it must not be overlooked that such errors, from their very nature, are apt to have a constant value, and are for that very reason the more liable to deceive. In the investigation just referred to, we showed that under constant conditions the composition of the crystals was definite and invariable, although they might contain — as dirt or impurity, very possibly — an excess of one or the other elementary substance when compared with the normal atomic proportion. Assuming, then, such an excess to be present in the substance selected for analysis, it is perfectly evident that a large error might exist in the determination of an atomic weight, although there was a close agreement in the analytical results. Such agreement therefore is in itself no proof of accuracy; and far less sharp results may be more trustworthy, if only it can be shown that the errors are properly distributed. Errors of the last class can be to a great extent eliminated by multiplying observations, while a constant error is only perpetuated thereby.

The special cause of constant error we have been discussing is one we have specially studied, and one therefore which we naturally select to illustrate the principle we have aimed to enforce. It is not, however, the only cause of constant error which tends to change the apparent ratio between the weights of the various elements of a compound, or the one whose influence is most to be feared in the determination of an atomic weight. Our knowledge of the chemical and physical relations of the materials analyzed, or of the circumstances connected with the processes employed, is in many cases as yet so far from perfect that to a certain extent we work as it were in the dark, and are liable to fall into errors from which only more accurate knowledge could protect our results. As will appear in the sequel, this truth has been forced upon us again and again during the course of the present investigation, and it has led us to bestow upon the work an amount of labor which is far greater than the importance of the results would seem to justify. The experience, however, has left with us a strong conviction on two important points; and we shall describe the several steps in our investigation more in detail than might otherwise seem necessary, in order that the reasons of these convictions may appear. In the first place, we are persuaded that in the determination of atomic weights it is just such constant errors as we have encountered that are almost solely to be feared, and that the mechanical perfection of our analytical methods is far in advance of our chemical knowledge of the materials and processes we employ. In the second place, we feel assured that no agreement, however close, of results obtained by the repetition of the same process, under the same conditions, gives any certain guarantee of the correctness of the atomic ratio which is sought to be established; and hence that in the present state of science no certain conclusions can be reached in regard to the validity either of Prout's Law or of other numerical relations between the atomic weights of the chemical elements.

We return now to our former position,— that in determining an atomic weight it is of the first importance to show that the compound analyzed is not only pure in the ordinary sense, but also that it contains the elements to be compared in atomic proportions. And, after carefully reviewing the whole subject in the light of our present knowledge, we have been led to the conclusion that the most satisfactory evidence on this point we can obtain is that furnished by a chemical reaction, in regard to which it can be shown that the two elements under comparison are transferred without loss or elimination of material from one combination to another. Now, satisfactory evidence of this kind may be adduced in regard to the method of determining the atomic weight of antimony employed by Schneider. The native sulphide of antimony which he analyzed dissolves in hydrochloric acid with the evolution of hydric sulphide, leaving no residue saving a minute amount of silicious gangue, which can be accurately estimated. If then sulphur is combined in atomic proportions with hydrogen in the gas evolved during this reaction, it must also have been combined in atomic proportions with antimony in the original compound; for otherwise the metathesis could not have taken place without an elimination of the excess of one or the other of the constituents. To this familiar fact, we can add still other evidence which makes the proof complete. In the first place, we have repeatedly verified the statement of other chemists, that when — as is always the case unless great care is taken — an excess of sulphur is precipitated by the action of hydric sulphide on solutions containing antimony, this excess, however small, is eliminated, and remains undissolved when the dried precipitate of Sb_2S_3 is dissolved in pure hydrochloric acid. In the second place, during our investigation of the zinc and antimony compounds above referred to, we prepared artificial crystals of antimony glance containing several per cent of antimony in excess of the normal proportion. These crystals were made by fusing the native mineral with an excess of metallic antimony, and pouring out the still melted mass from the crucible, after a portion had crystallized. But although the crystals were extremely brilliant, and had at least this outward characteristic of a definite compound, yet, when dissolved in hydrochloric acid out of contact with the air,* the excess of metal was left undissolved in the condition of a fine powder.

For the reasons we have stated, we have always had great confidence in the results of Schneider, while on the other hand we have felt that both in the case of the oxide of antimony weighed by Dexter, and the chloride of antimony used by Dumas, we could not from the nature of the case have any satisfactory evidence that the material analyzed had the exact atomic composition assigned to it. The antimonic oxide is a perfectly amorphous and inert powder, in regard to which we have neither the evidence of physical properties nor of chemical reactions on the point in question. Moreover, our knowledge of the circumstances under which it is formed would lead us to suspect an admixture of a lower oxide in the product, which would be protected from oxidation (during the ignition) by the surrounding mass, and protected to the same degree under the same conditions. Again, in SbO_2 the weight of the oxygen is only about one-fourth of that of the antimony, so that a very small variation in the weight found would make, in the result, all the difference which is in question. In chloride of antimony, on the other hand, the amount of chlorine is nearly equal to that of the antimony; and for this reason, as well as because chlorine can be so accurately determined, this substance would seem at first sight to be the best adapted of all others for determining the atomic weight of antimony. Our experience, however, has not confirmed this first impression; for although we have found no difficulty whatever in obtaining the material beautifully crystallized and free from such impurities as we should ordinarily look for, yet it is so wonderfully hygroscopic and liable to alteration that we have not as yet succeeded in preparing it for analysis under such conditions that we could feel assured that it was wholly free from moisture or the resulting oxichloride of antimony; and although with sufficient labor and ingenuity the difficulties in the way could undoubtedly be overcome, yet a substance which must be guarded with such precautions is not the best adapted for determining an atomic weight. When, therefore, we had devised the method of precipitating metallic sulphides, described in the previous volume of this series,* and found that by its means, and also by boiling the liquid, as suggested by Mr. S. P. Sharples,† we could precipitate sulphide of antimony in a condition apparently peculiarly well adapted for accurate determination, we conceived the idea of reversing Schneider's method, and verifying his result by a synthesis of the same material which he analyzed. The event, however, proved that we had undertaken a work of far more difficulty than we anticipated. We have met unforeseen obstacles at almost every step of our investigation, and have had constant occasion to indorse, with Kessler, the opinion of Berzelius quoted above.

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* Antimony is wholly insoluble in hydrochloric acid out of contact with the air.

In the preparation of pure metallic antimony, we were greatly guided by the experience of Mr. Dexter;‡ and our several products must have been very similar to his, as the following determinations of the specific gravities of the different buttons show. The observed values were reduced to 4° C., on the assumption that the coefficient of cubic expansion for antimony between 0° and 100° C. is for each degree 0.000033, as observed by Kopp. The letters here given will be used throughout the table to designate the various specimens. As might be supposed, the specimens were prepared at different times and at different stages of the investigation, but the results are united here for the convenience of comparison and of reference.

SPECIFIC GRAVITIES OF BUTTONS OF PURE METALLIC ANTIMONY.

Observations of J. P. C., Jr.		Observations of W. Dexter.	
A	6.7025	b	6.7087
B	6.7036	c	6.7026
C	6.6957	c	6.6987
D	6.7070	d	6.7102
E	6.7022	e	6.7047
F	6.7023	e	6.7052
Mean	6.7022	Mean	6.7050

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* On a New Mode of Manipulating Hydric Sulphide, vol. xii. of these Proceedings, p. 113.

† American Journal of Science and Arts, Second Series, vol. l. p. 248.

‡ Poggendorff's Annalen, Band 100, Seite 564 (*loc. cit.*).

Specimen A was prepared from potassic antimoniate (marked Robiquet and Pelletier, *pur*). A solution of the salt in water was filtered into a similar hot solution of sodic carbonate. The precipitated sodic antimoniate was thoroughly washed with hot water, then dried, and reduced with potassic cyanide. Lastly, the regulus was remelted, and kept in fusion for several hours under its own oxide. Specimen B was prepared, in a similar way, from potassic antimoniate (marked Rousseau Frères, *pur*). Specimen C was prepared from commercial antimony. The metal was first oxidized, and the oxide boiled with an excess of pure nitric acid. The oxide was afterwards repeatedly washed with boiling water, and, when dried, was reduced with potassic cyanide. A third of the resulting metal was again oxidized with nitric acid; and this oxide having been first mixed with the rest of the metal, previously pulverized, the mixture was kept melted for a long time in a covered crucible. The purified metal thus obtained was again fused in a porcelain crucible, under its own oxide, for several hours. Specimen D was the residue of C, — part having been used for casting bullets, — again fused for several hours under its own oxide. Specimen E was prepared by the process described above from potassic antimoniate, made by ourselves from commercial antimony. It was fused for four hours under its own oxide. Specimen F was prepared by Liebig's well-known process, and further purified by fusing the regulus for twenty-nine hours under its own oxide. It is unnecessary to add that these fusions were all made in porcelain crucibles, and that only the purest reagents were employed in the various processes, except only in the early stages of the preparation of C and F. These last preparations were both used for casting the antimony bullets with which the solutions of the metal were reduced before precipitating with H_2S, as will be described hereafter.

We give, in a parallel column with our own results, the specific gravities, determined by Mr. Dexter, of the specimens of pure antimony prepared by him, and used in his determinations of the atomic weight of this element. It will be noticed that the agreement is very close, the mean of his results not differing from that of ours as much by one-half as either set differ among themselves. These differences are evidently owing to slight variations in homogeneity, due to the crystalline structure of the metal. This is made probable by the following results which we obtained with specimen F, prepared as described above by Liebig's process. This was repeatedly fused in its own oxide, and the specific gravity taken at each stage of the process.

SPECIFIC GRAVITY OF SPECIMEN F.

After five hours' fusion, additional . . .	6.7050
" " " " " . . .	6.6977
" " " " " . . .	6.6959
" " " " " . . .	6.6840
" nine " " " . . .	6.7022
" several " " " . . .	6.7012

Such irregular variations as these could obviously not be referred to a gradual change in the purity or composition of the metal.

Of the several processes employed for purifying antimony, the last is the easiest, yields the largest product, and is as effectual as either of the other two. The long fusion of the metal under its own oxide can in no case be avoided, as this is the only sure method of removing the last traces of iron, at least so far as our experience extends. In repeating the work, however, we should prefer to start from antimonious chloride, prepared from antimony glance. This can easily be purified by repeated distillation, and is absolutely free from arsenic, which cannot always be said of the metal obtained by Liebig's process.

To bring a *given quantity* of antimony into solution, so that the whole shall be left in its condition of lowest quantivalence, and therefore be precipitated by H_2S as antimonious sulphide, without the least excess of sulphur, we found to be a very difficult task; nor have we succeeded invariably in effecting this result, except by the indirect method hereafter described. Finely powdered antimony dissolves readily when digested with a concentrated solution of tartaric acid, to which a few drops of nitric acid have been added; but from such a solution, even after the remaining nitric acid has been carefully neutralized, free sulphur is always precipitated with the sulphide of antimony; and, although we tried various ways, we were unable to find any process by which this effect could be wholly prevented. Very singularly we obtained the best results by acting on antimony directly with nitric acid, under such regulated conditions that the metal was wholly converted into antimonious nitrate, and subsequently removing the excess of acid by slow evaporation at a temperature below 100° C. If the conversion is successfully accomplished, then, on adding to the crystalline residue a strong solution of tartaric acid, this residue is at once dissolved, giving a perfectly clear solution, in which the metal is almost wholly, if not entirely, in its lower condition of quantivalence. The difficulty consists in regulating the action of the nitric acid. We succeeded best by using acid of the sp. gr. 1.35, which we added in large excess to the powdered antimony, previously moistened with water, and as soon as, on gently heating the beaker, the action began, we kept the mass in constant agitation, so as to prevent the temperature from rising too high at any point. If during this reaction, or during the subsequent evaporation of the excess of acid, the dry nitrate is heated even to 100°, decomposition ensues, and higher oxides are formed, which are insoluble in tartaric acid. Indeed, this is probably the order of the change even during the violent action of nitric acid on antimony, when antimonic oxide, Sb_2O_4, is the chief product, antimonious nitrate being in all cases formed first, and subsequently decomposed. Unfortunately, we could not always succeed in obtaining a clear solution in this way; and the loss of pure material occasioned by the unsuccessful trials was so considerable that we soon abandoned the method. Still, it was capable of yielding good results, as the determinations given below will show.

The rest of the process was as follows: The tartaric acid solution having been brought to the proper degree of dilution, — about 250 cubic centimetres to each gramme of antimony, — the free acid was first carefully neutralized with pure caustic soda. A small amount of hydrochloric acid was next added, and the antimony precipitated with H_2S. In these, as well as in all the subsequent determinations, the precipitation was conducted thus: The solution having been made in a large Erlenmeyer flask, and this firmly supported over a gas lamp, a current of washed carbonic dioxide gas was conducted into the liquid from a suitable generator, the glass tube conducting the gas reaching to the bottom of the flask, so that the gas might bubble up through the whole column of liquid. The atmospheric air having been thus completely expelled, then, through a second glass tube similarly arranged, an excess of a supersaturated solution of H_2S was drawn into the vessel from the fountain or siphon described in the paper above referred to. The temperature of the mass was now slowly raised to the boiling point, and the liquid actually boiled for ten or fifteen minutes. The current of carbonic dioxide, continually passing meanwhile, not only excluded the air, but kept the mass in constant agitation and prevented "bumping."

The precipitate thus forms in a granular condition excellently well adapted to bear without alteration the processes of washing and filtering which follow. It does not adhere to the tubes or sides of the flask, and unless there is some oxidizing agent in the solution it does not contain the least trace of free sulphur. The precipitate was washed with boiling water, and as it settles rapidly and quite compactly it was easily washed in the same flask in which the solution was made and the precipitate formed. The precipitate usually occupied about one-tenth of the volume of the flask, and after drawing off the filtrate our rule was to fill the flask five times with boiling water. We were sometimes troubled by the breaking up of the precipitate and the washing of it through the filter, under such prolonged treatment. As is well known, this can easily be prevented by acidifying the wash water, however slightly, with hydrochloric acid, or by an excess of H_2S; but as the last was liable to oxidation, and the first still more objectionable, for reasons which will soon appear, we employed for the same purpose a solution of carbonic dioxide, and this, as we found, effectually prevented the disintegration. If by not settling promptly the precipitate showed any signs of breaking up, we passed a current of CO_2 through the hot water before it was used in the last one or two washings.

In all the later determinations we collected and dried the precipitate in a large platinum crucible by the method of reverse filtering illustrated and described at length on page 124 of the previous volume of these "Proceedings." But, in the two determinations made by the method just described, the precipitate was collected and dried on the porous earthenware filtering cones described by Prof. C. E. Munroe.[*] The cones with the precipitate were first dried at 100°, and then heated to 220°C. in an atmosphere of carbonic dioxide. Between 210° and 220°, depending somewhat on its condition, the red sulphide of antimony changes suddenly into the gray modification, and in this last condition the product was weighed. In each of these two determinations, on dissolving the gray sulphide in boiling hydrochloric acid, out of contact with the air, a considerable quantity of black residue was obtained. This contained only a very small amount of free sulphur, but consisted chiefly[†] of carbon, resulting from the charring of the small quantity of tartaric acid, which, as we find, sulphide of antimony invariably carries down with it when precipitated under the conditions here described, and which cannot be removed by any amount of subsequent washing. This charring takes place at the moment when the red is converted into the gray sulphide, and is undoubtedly caused by the heat evolved during this process. That at this point the charring is perfect, and that after the conversion no tartaric acid is left undecomposed, we have proved by several times heating the resulting gray sulphide to 300°, and ascertaining that it underwent no additional loss of weight. As is well known, small quantities of tartaric acid are completely decomposed at this last temperature.* In the two determinations we are discussing, the carbonaceous residue was collected on a weighed filter, and its weight subtracted from the total weight of the dried precipitate. All the remainder was pure Sb_2S_3; and, from its weight and that of the antimony used, the ratio between the atomic weight of sulphur and that of antimony was very easily calculated. The results were as follows: —

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[*] American Journal of Science, May, 1871.

[†] In our preliminary experiments, traces of both lead and copper appeared in this residue, which we traced at once to the tartaric acid "*purissimum*" used, and we found it impossible to procure from chemical dealers tartaric acid free from these impurities. We were obliged, therefore, to purify ourselves all the acid we employed.

First Determination.

Weight of antimony taken	2.0554	grammes.
Weight of precipitate dried at 240° . .	2.8878	„
„ carbonaceous residue. . . .	.0147	„
„ Sb_2S_3	2.8731	„
„ antimony as above	2.0554	„
Corresponding weight of sulphur . . .	0.8177	„
Hence when $S = 32$, then $Sb =$ . .	120.6	„

Second Determination.

Weight of antimony taken	2.0346	grammes.
Weight of precipitate dried at 240° . .	2.8513	„
„ carbonaceous residue	0.0073	„
„ Sb_2S_3	2.8440	„
„ antimony as above	2.0346	„
Corresponding weight of sulphur . . .	0.8094	„
Hence when $S = 32$, then $Sb =$ . .	120.6	„

It is evident that these results, so far as they go, very greatly tend to confirm the value $Sb = 120.3$ obtained by Schneider; and, in the light of the knowledge we have since obtained, the reason that our first results were somewhat higher than his is perfectly evident. Whenever sulphide of antimony is precipitated under the conditions we have described, it always carries down with it, not only a small amount of tartaric acid, but also a very appreciable quantity of oxichloride of antimony, $SbOCl$, which, like the impurity first named, cannot be removed by washing the precipitate. The molecular weight of $SbOCl$ (171.5) differs but slightly from the *equivalent* weight of Sb_2S_3 (168), so that, were the precipitate dried at 100°, a small admixture of this compound would not sensibly affect the total weight. But, as soon as the temperature reaches 170°, this oxichloride begins to be decomposed, $SbCl_3$ volatilizes, and the more stable oxichloride $Sb_4O_5Cl_2$ is formed. At a low red heat, this last compound is also in its turn decomposed, still more $SbCl_3$ escapes in vapor, and the final residue is antimonious oxide in a crystalline condition. These changes may be represented thus:—

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* Gmelin's Handbook of Chemistry, Cavendish Edition, vol. x. p. 209.

1st. $5\ SbOCl = Sb_4O_5Cl_2 + SbCl_3$.

2d. $3\ Sb_4O_5Cl_2 = 5\ Sb_2O_3 + 2\ SbCl_3$.

At least, these are the reactions when $SbOCl$ is heated by itself in an atmosphere of CO_2, as we shall show further on in this paper. In the presence of a large mass of Sb_2S_3, containing also a trace of organic matter, these effects are undoubtedly somewhat modified and have not been exactly traced. But whenever a precipitate of Sb_2S_3 formed as we have described is heated much above 150°, it yields a white sublimate; and this sublimate, which we have repeatedly tested, consists chiefly of antimony. At times it was pure chloride of antimony which crystallized on the walls of the glass tube used in the experiment; but, when very small in amount, the sublimate was an amorphous white powder, which appeared like oxichloride of antimony, and which may have been formed by the action of a minute quantity of chloride of antimony on the products of the decomposition of the equally small amount of tartaric acid also occluded by the precipitate. It has long been known that precipitated sulphide of antimony dried at 100° loses weight when heated to a higher temperature, and it has generally been assumed that this loss was due to hygroscopic water. But red sulphide of antimony, when precipitated as we have described, is not in the least hygroscopic and can be dried perfectly at 100°. At least, we have never been able to obtain evidence that, after being thoroughly dried at this temperature, it ever contains water as such; and we are confident that the loss of weight is due solely to the causes we have assigned. At the time the first two determinations were made, these facts were not known, and no allowance was made for the loss of chloride of antimony which must have been incurred. This loss fully accounts for the difference between the value we obtained, 120.6, and 120.3, that of Schneider.

It was at this stage of the investigation that we presented a preliminary notice of our results to the American Academy of Arts and Sciences, at the meeting of June 10, 1873. After this, the work was interrupted for more than two years, and was not resumed until the autumn of 1875. Meanwhile, we had perfected the process of reverse filtering above referred to, and devised a more certain, although indirect method of bringing into solution, in its lower condition of quantivalence, a known weight of antimony. This last consisted, firstly, in dissolving a weighed portion of antimony in hydrochloric acid with the least possible addition of nitric acid; secondly, in reducing the solution thus obtained, by boiling it over bullets of pure antimony, determining from the loss of weight of the bullets the amount of metal dissolved. Evidently, the mass originally taken, plus the amount dissolved from the bullets, gave the weight of antimony used in each determination. Several important facts were observed in connection with each of these steps.

1st. It is usually stated that antimony is only very slightly acted on by pure hydrochloric acid, even when concentrated and boiling, but that it readily dissolves on the addition of only a very small amount of nitric acid.* We found that, when wholly protected from the air or oxidizing agents, pure antimony not only does not decompose pure hydrochloric acid, but also that in contact with the air the smallest amount of nitric acid will determine the solution of an indefinite amount of the metal. Assuming that the product of the reduction of the nitric acid is wholly NO, and therefore that the smallest amount theoretically required for the reaction would be represented by the expression, —

$$\mathrm{Sb} + (3\mathrm{HCl} + \mathrm{HNO_3} + \mathrm{Aq}) = (\mathrm{SbCl_3} + 2\ \mathrm{H_2O} + \mathrm{Aq}) + \mathrm{NO},$$

then 40 parts of antimony would require, in addition to an abundant supply of hydrochloric acid, 21 parts of HNO_3 for its complete solution. The pure nitric acid which we used had a specific gravity of 1.355, and contained therefore about 54% of HNO_3. The same acid diluted with nine times its volume of water, to form what we will call the decim acid, contains only 5.4% of HNO_3; and hence one gramme of antimony, according to our reaction, ought not to dissolve, if less than 10 cubic centimetres of this decim acid were used. Now, in one experiment, of which we have the details on record, 5 grammes of finely powdered antimony were treated in an open flask with 50 cubic centimetres of strong and pure hydrochloric acid, and to this only 1 cubic centimetre of the decim acid was added. The flask was placed in a warm place (30° C.), and frequently shaken. After a short time, the acid became colored reddish-yellow, and the chemical action began. As soon as it ceased, the materials in the flask were shaken together, when the solution became as colorless as water. But on standing in the air the color rapidly returned, and we observed that it always spread from the surface of the liquid downwards. These phenomena were repeated again and again, until after many days the whole of the antimony dissolved. According to our reaction, the 5 grammes of metal should have required 50 cubic centimetres of acid, so that the effect was obtained with only one-fiftieth of the amount indicated by this theory. The action is probably due to the NO, which remains in solution, and in conjunction with the oxygen of the air decomposes the hydrochloric acid, perhaps thus, —

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* Author's Chemical Philosophy, p. 265.

$$2\,NO + 4\,HCl + O_2 = 2\,NOCl_2 + 2\,H_2O,$$

then

$$Sb_2 + 3\,NOCl_2 = 2\,SbCl_3 + 3\,NO;$$

and afterwards these reactions are repeated indefinitely. But, whether they represent the precise order of the chemical changes or not, there is no question that here, as in the complex reactions of the sulphuric acid chambers, the nitric acid or its products acts as a carrier, and that the oxygen which combines with the hydrogen of the hydrochloric acid comes chiefly from the atmosphere. In practice, we usually used, for dissolving 2 grammes of antimony, 30 cubic centimetres of strong hydrochloric acid and 5 cubic centimetres of the decim nitric acid; and, although this is only one-fourth of the amount of nitric acid required by the formula generally accepted, it is sufficient to determine a very energetic chemical action, during which a large part of the nitric oxide escapes. The action does not generally come on for some time (fifteen or twenty minutes), although it can be hastened by placing the flask in a warm place. But otherwise we did not heat the acid until almost the whole of the antimony was dissolved. When only a few centigrammes of metal remained undissolved, we connected the flask with a reversed Liebig's condenser and raised the contents to the boiling point. The effect of this was to arrest the chemical action by driving off the nitric oxide; and, if the point was rightly chosen, the small residue of antimony, in passing into solution, would reduce almost all of the higher chloride of antimony which had been previously formed. We found it better, however, that some of the higher chloride should be left unreduced rather than that the least trace of antimony should remain undissolved. In the last case, it was necessary to add additional nitric acid, which made trouble at the next stage of the process; and the fact repeatedly observed, that under these circumstances a few milligrammes of antimony will not dissolve in a large excess of hydrochloric acid, even after prolonged boiling, is a sufficient proof of the statement made above,—that pure antimony will not decompose pure hydrochloric acid, unless some oxidizing agent is also present.

2d. Having thus brought a weighed amount of antimony into solution, so that almost the whole was in the condition of antimonious chloride, we completed the reduction by boiling the acid solution over bullets of pure antimony. These bullets were cast in a bullet-mould, and afterwards turned in a lathe, so as to secure a perfectly clean, smooth, and compact surface. Two or three of them were next weighed together in a platinum tunnel, supported vertically by a light glass stand, on the pan of the balance. From the tunnel they were transferred one by one, without touching with the fingers, to the flask containing the solution of antimony made as just described. The flask was then again connected with the reverse condenser, and the liquid boiled; while through a second opening in the cork a slow current of carbonic dioxide was caused to flow through the apparatus. The boiling was continued until the reduction was complete, and the point was indicated by the circumstance that, when it is reached, the solution, at first having a light straw color, becomes perfectly colorless. The original color is caused by the presence of an exceedingly minute amount of iron, which it is almost impossible to keep out of the solution. Indeed, the very dust of the atmosphere will impart enough for the purpose; and this color under these conditions is as delicate a test for iron as that caused by potassic sulphocyanide. It served here as a very useful indicator, but it was no sign whatever of any appreciable amount of impurity, either in the metal or the acid. Another indicator which can be used, although not nearly so conveniently, is the well-known solution of potassic iodide in starch paste, the acid solution of chloride of antimony giving with this reagent the familiar blue color before, but not after, it has been reduced.

It has long been known that the vapor from a boiling solution of antimony in hydrochloric acid carries off a portion of the chloride of antimony. We have found that this is true only when the solution is quite concentrated: that the amount which escapes rapidly diminishes as the solution is diluted, and that it soon becomes wholly imperceptible. Still, as in our experiments the boiling was frequently greatly prolonged, we guarded against any loss from this cause by using the reversed condenser, as above described. The time required for the reduction varied very greatly under different conditions. It was seldom finished in less than an hour, and the process frequently required several hours. The boiling was stopped as soon as the pale color of the solution was perfectly discharged; but, while the flask was cooling, the current of carbonic dioxide was steadily maintained. When cold, a measured portion of a concentrated solution of tartaric acid (containing generally from ten to fifteen grammes of the crystallized acid) was added to the flask, and the contents were then at once transferred to the large Erlenmeyer flask in which the antimony was to be precipitated. The transfer was accomplished very easily and perfectly in the following way.

Into the Erlenmeyer flask was first poured about 500 cubic centimetres of water strongly charged with carbonic dioxide; and then the platinum tunnel, on which, as we have stated, the antimony bullets had been weighed, and which had been carefully protected meanwhile, was placed in the mouth of the large flask. As the solution was now poured in from the smaller vessel, the bullets were of course caught by the tunnel, and the aërated wash water, which also passed through the tunnel, served to wash the bullets as well as the glass. The tunnel and its contents were then dried and weighed, and the loss from the previous weight gave accurately the amount of additional metal which had passed into solution during the process of reduction. It will be noticed that, during the whole process, the balls were never touched with the fingers, or brought in contact with any object by which their weight could be in the least altered. It was found however, to be essential to the success of the method that not more than a few decigrammes of metal, at most, should be dissolved from the balls; for, otherwise, the surfaces became disintegrated, and liable to abrasion. Hence, the objection to using an excess of nitric acid in dissolving the original quantity of antimony.

The precautions here described may seem unnecessary to those who are not familiar with the fact that a solution of antimony in hydrochloric acid oxidizes with very great rapidity in the air, — fully as rapidly as the solution of a ferrous salt. A solution reduced as we have described, which has at first no action on the iodized starch paste, will strike the blue color after it has been exposed to the air for only a few minutes. This property of an acid solution of antimonious chloride is mentioned by Dexter, in the paper already referred to, but we were wholly surprised by the energy of the action. By means of it, antimony can be dissolved in hydrochloric acid without the aid of nitric acid, or of any other oxidizing agent save the air, if only a certain amount of antimonious chloride has once been formed. When, after exposure to the air, the solution is boiled over pulverized antimony, the solution is reduced, and a further portion of the metal enters into solution. After a second exposure, the same process can be repeated, and so on indefinitely. The process is very slow and tedious, but, in one experiment, we succeeded in bringing into solution in this way several grammes of antimony.

The method of precipitating the antimony after the solution was thus prepared has already been described (page 14). It is important, however, to add a few additional facts, which we observed in regard to the two impurities which the precipitated sulphide of antimony occludes; namely, tartaric acid and oxichloride of antimony. In the first place, we found that the relative amounts of these occlusions might be varied indefinitely by changing the relative proportions of tartaric and hydrochloric acids in the solution in which the precipitate was formed. As is well known, the antimony cannot be kept in solution without a certain excess of one or of both of these reagents. In proportion as the amount of one is diminished, that of the other must be increased; and we made a series of experiments to determine what were the minimum quantities required under different conditions. We made determinations in which the antimony was held in solution by hydrochloric acid only; and we found that, in that case, as much as one part of strong acid was required to five parts of water. We thus, of course, wholly avoided the occlusion of tartaric acid; but the amount of oxichloride carried down was so great that the results were worthless. On the other hand, in some more recent analyses, in which we began with pure chloride of antimony, we used only tartaric acid, and, in this case, although the carbonaceous residue was at times large, there was no loss by sublimation. Of the two occlusions, the tartaric acid is by far the least objectionable. This, as we have seen,* is wholly charred when the red sulphide of antimony is converted into the gray modifications, and the carbonaceous residue can be exactly estimated. In the case of the oxichloride of antimony, on the other hand, it is impossible to obtain any satisfactory control either over the chloride of antimony which escapes or the oxide which is left behind. Unfortunately, in starting from metallic antimony, we cannot avoid a large excess of hydrochloric acid in the solution, and must therefore expect more or less oxichloride of antimony in the precipitate. Assuming therefore that 15 cubic centimetres of strong hydrochloric acid were taken to dissolve every gramme of antimony, which was as little as could conveniently be used (but from this the greater part of the HCl gas was expelled during the subsequent boiling), it became, in the second place, an object to determine how much tartaric acid was required to hold the antimony in solution, and to reduce to a minimum the occlusion of oxichloride of antimony by the precipitated sulphide. With this view the following experiments were made. Four solutions were prepared by dissolving in each case 2 grammes of antimony in 30 cubic centimetres of hydrochloric acid, and reducing the solution as above described. To the first of these were added 5 grammes of tartaric acid, to the second 7 grammes, to the third 10 grammes, and to the fourth 20 grammes. They were then each diluted with water to one litre. In the first, oxichloride of antimony was precipitated at once. The other three remained clear; but, in the second and third, crystals of oxichloride formed on standing over night, and in the inverse proportion to the amount of tartaric acid added. On subsequently heating to boiling, the same crystals were precipitated even in the last, and the amount in the others very greatly increased. This crystalline precipitate formed as the solution was heated, and was deposited like acid tartrate of potassium wherever the glass rod touched the sides of the flask. From the first and second solutions almost the whole of the antimony was thus separated.

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* Page 15.

These crystals were analyzed, and found to be the oxichloride $Sb_4O_5Cl_2$. They will be described in a future paper. It is evident from these experiments that, under the conditions we have given,* from 5 to 10 grammes of tartaric acid to each gramme of antimony are required to prevent the formation of oxichloride of antimony, even in a cold solution; and our experience shows that, even with the larger amount, a perceptible although very small quantity of oxichloride is occluded by the precipitate formed with hydric sulphide. Moreover, our experience indicates that, with an intermediate amount, the loss and gain resulting from the causes we have mentioned are more or less closely balanced. By preserving as nearly as possible identical conditions, and taking, for example, to every 2 grammes of antimony 30 cubic centimetres of hydrochloric acid and 15 grammes of tartaric acid, it was found possible to obtain results closely agreeing not only with each other, but also with what we finally concluded was the most probable value of the atomic weight of antimony. But, during the investigation by which these facts were developed, we made many determinations with varying proportions both of hydrochloric and tartaric acid, in which the error arising from an excess of one or the other must have had its full effect. All these determinations not obviously defective are recorded below; and although a closer agreement would appear, were only those determinations selected which were made after the more accurate knowledge was obtained, and therefore under more favorable conditions, yet we feel much greater confidence in the result obtained by taking the mean of all, since in this mean the errors must be to a great extent, if not perfectly, balanced.

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* In solutions of the same strength which have not been perfectly reduced, although containing but little of the higher chloride, the oxichloride does not appear to form nearly so readily; but we have made no quantitative experiments on this point.

We were for some time in doubt in what condition the sulphide of antimony ought to be weighed, in order to obtain the most accurate results. Our final judgment was that the errors already referred to would best be balanced, while others would be avoided, by weighing the sulphide, after it had been dried, at from 180° to 200°, but before it was actually converted into the gray sulphide. This conversion takes place between 210° and 220°, varying to that extent in different cases. The change, as we infer, is attended with a sudden evolution of heat, and the action is quite violent. Small particles of the material are frequently projected from the vessel, and we sometimes noticed that the surface of the platinum nacelle became coated with the familiar sublimate of sulphide of antimony. If there was oxichloride in the precipitate, there may be an additional volatilization of chloride of antimony at this time; but the main loss, as we have constantly observed, takes place before the point of conversion is reached. We therefore concluded that more trustworthy results could be deduced from the weight of the red sulphide dried, as we have described, than from that of the gray; and, as will be seen, this judgment was fully confirmed by subsequent experiments on the haloid compounds. We have, however, in all but two instances weighed the sulphide in both conditions, and we give the results of both weighings; and on comparing these results in determinations 8 to 13 inclusive of the table on pages 36–7, which were made under the nearly identical conditions we have above indicated, it will be seen that the differences are far smaller with the red sulphide than with the gray, which shows conclusively that additional causes of error must have affected the last weights, — another circumstance which sustains our judgment.

In the first twelve determinations we did not estimate the amount of the carbonaceous residue, which is assumed to be balanced by the loss of chloride of antimony, although we always tested the purity of the sulphide of antimony by dissolving it in hydrochloric acid, as described. In determination numbered 13, by some chance concurrence of favorable conditions, we succeeded in precipitating the antimony without the usual occlusion of oxichloride, although we used a large excess of hydrochloric as well as of tartaric acid. In this case, there was no evidence of sublimation nor loss during conversion, but a proportionally large carbonaceous residue, which was deducted from the weight of the sulphide; and the result of this determination, as will be seen, still further corroborates our conclusion. The same is true of some analyses of chloride of antimony made more recently, in which we dissolved crystallized chloride of antimony in a concentrated aqueous solution of tartaric acid, without using any excess of hydrochloric acid. In these cases also, the drying of the precipitate, and the conversion from the red to the gray modification, were attended with no appearance of sublimation. Were we to repeat the investigation with our present knowledge, we should follow the indications of these last analyses; and instead of attempting to make the two chief errors as small as possible, and balance them, we should seek to remove from the solution all the free hydrochloric acid, and thus eliminate the error due to the occlusion of oxichloride.* It would then, of course, be necessary to determine in all cases the carbonaceous residue, which might however be very large, without impairing the accuracy of the result. Still, our experience with these antimony determinations would lead us to fear that we might thus raise up as many hindrances as we avoided, and the determination we have given as No. 13 is sufficient for all purposes of comparison.

Before giving the results, it only remains to describe the manner in which the precipitates were dried, tested, and weighed. After that, by the method of reverse filtering, the precipitate had been washed, and collected in a large platinum crucible, as described on pages 14 and 15 of this article, it was dried in an air bath, at a temperature varying from 100° to 130° C. in different cases; and, with time enough, the lower temperature seemed to be equally effective. Thus in one case we have the following weights recorded:—

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* Our experiments also indicate that, even in presence of a large excess of hydrochloric acid, the occlusion of oxichloride can be prevented by using a very large excess of tartaric acid. It was under these conditions that the determination No. 13 was made. Of course, the occlusion of tartaric acid is then large; but, as shown, this need not impair the accuracy of the result.

Weight of Sb_2S_3 dried in a steam-chest . . .	2.1700	grammes.
„ same after 30 minutes at 150° . .	2.1685	„
„ „ „ additional .	2.1685	„
„ „ conversion at 210° . .	2.1677	„

By drying, as is well known, the precipitate shrinks to a very small volume. A small portion was then taken and dissolved in hydrochloric acid, in order that we might be sure no free sulphur was present; but so effectual were the precautions against oxidation we have described, that after our process was perfected we obtained in no case the least trace of residue. The larger part of the dried precipitate was next transferred to a platinum nacelle; and, the weight of this portion having been exactly determined, the nacelle was introduced into a glass tube about three-fourths of an inch in diameter. The portion of the tube holding the nacelle was heated by an air bath, through which the tube passed and was tightly held. This air bath was made of sheet iron. It had a double bottom, and a tubulature on one side for a thermometer. The cover consisted of a thin sheet of mica, through which the nacelle could be seen within the tube, and every change accurately observed. The bath was heated by a common gas burner, and the temperature regulated by regulating the flow and pressure of the gas. During the heating, a slow current of hydrogen was passed through the tube. This gas, made from oil of vitriol, zinc, and water, in a large germinator, was first purified by passing through solutions of caustic potash and nitrate of silver, and afterwards dried by sulphuric acid and chloride of calcium. The gas entered and passed out through tightly fitting corks, and the glass exit tube was made small in order that the least condensation of moisture on its walls might be the more apparent.* In this simple apparatus, the dried precipitate was maintained at a temperature between 180° and 200°, until the weight was constant. The nacelle was then again returned to the tube and heated to 210°, or until the conversion of the red sulphide of antimony into the gray modification took place, and the weight again taken. Lastly, from the loss of weight thus observed, the correction to be subtracted from the first weight of the whole precipitate was easily calculated. It is evident that with this apparatus we were enabled to collect and examine the products sublimed during the heating, and form a very accurate estimate of the relative amounts under different conditions; and it was from the phenomena thus observed that we deduced the inferences which have already been stated. One or two additional remarks, though in part a repetition, are here important.

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* The apparatus is the same as that subsequently used for the sublimation of bromide and iodide of antimony, and figured on page 57, with this exception, that the large adapter which there serves as a receiver is replaced by a small glass tube, as described above.

1. Although, during the course of the investigation, the experiment was repeated a great number of times, yet in no case, even when the precipitate was dried at 100°, was there the least deposition of water on the walls of the exit tube. Were there as much hygroscopic water present as has been usually supposed, it must have shown itself under these circumstances. Of course, a very minute amount would escape detection, and we sought to obtain more positive evidence, both by means of dried sulphate of copper and also by a chloride of calcium tube; but the evidence of the first was wholly negative, and from the slight and irregular variations of the last no positive indications could be drawn. In some cases, the weight of the chloride of calcium tube was obviously affected by the vapor of chloride of antimony; while in other cases, as in the example cited on page 25, the accidental variations in the weight of the tube were greater than the quantity, if any, we sought to estimate. During the charring of the occluded tartaric acid, of course some water must be formed, and the decomposition was made very evident by the familiar empyreumatic odor, which was always plainly perceptible; but the action was too complex and irregular, and the amount of the products too small, to admit of any trustworthy estimates by the usual methods.

2. The change of sulphide of antimony from the red to the gray modification is very sudden and striking. The temperature at which it takes place varies between 210° and 220°, depending on conditions which we could not wholly trace. It begins at some one point, and then spreads very rapidly through the mass. During the change, as has been stated, small particles of the material are sometimes projected from the nacelle; and there is not unfrequently evidence that at some points of the mass the temperature must have risen high enough to volatilize the sulphide of antimony. Frequently also at this time an additional amount of white sublimate was formed, and the peculiar empyreumatic odor again perceived. Under these circumstances, there was always a decided loss of weight. But at other times, especially

when the precipitate contained no oxichloride of antimony and had been heated for several hours to a temperature near the point of conversion, the change was attended with no loss of weight which was appreciable, as, for example, in the experiments cited on page 25, and in the determination numbered 13 below. After conversion, as we found in several experiments, the gray sulphide may be heated to 250° or even 300° without further loss.

The change of condition just described is not attended, however, with any very marked amount of condensation. We made the following determinations of the specific gravity of the sulphide of antimony in the two conditions. The specimens were weighed in alcohol of known density, and the specific gravity referred to water by calculation. An air pump was used to remove any entangled air.

RED SULPHIDE OF ANTIMONY DRIED AT 180°.

1. Specific Gravity determined at 26°.7 = 4.226.
2. " " " " 23°. = 4.223.

GRAY SULPHIDE OF ANTIMONY* CONVERTED AT 210°.

1. Specific Gravity determined at 28° = 4.288.
2. " " " " 27° = 4.289.

The published determinations of the specific gravity of the native antimony glance give values varying between 4.52 and 4.75; and Heinrich Rose gives † for the specific gravity of the artificial product made by melting together the constituents the value 4.614, and for the same, after pulverizing, 4.641. He also gives for the precipitated sulphide the value 4.421. It is evident, therefore, that the value varies, but in the above determinations the comparison of the two modifications were made under as nearly as possible identical conditions. Possibly, the decomposition of the occluded tartaric acid, producing a more or less spongy condition in the mass, may be the cause of the observed differences.

In the later determinations, when for the reasons we have stated the occlusion of tartaric acid was large, the gray sulphide, after having been weighed, was always dissolved in hydrochloric acid, and the carbonaceous residue estimated. The solution, having been mixed with tartaric acid and diluted with water, was filtered, and the residue collected on one of the smallest paper disks used in the process of reverse filtering. The disk was then dried and reweighed. The constancy of weight with these paper disks was very remarkable; and it may give greater confidence in the accuracy of our method to add here a few comparisons of the weights of the larger-sized disks which were used in collecting the sulphide of antimony itself, taken before and after they had been used. The first column gives the original weight of the disks, which were first dried at 120°, and then kept in an atmosphere dried by sulphuric acid. The second column gives the weight of the same after it had been used in filtering, and taken from the crucible after the first weighing with some of the dried precipitate adhering. We have only one weight of this kind recorded; but this will show how little of the precipitate adheres to the filter. The third column gives the weight of the same paper disk, after washing first with sulphide of ammonium, then with water, and drying.

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* Same as obtained in determination of table, pages 36–7, No. 8.

† Poggendorff, Annalen, lxxxix. 122.

	I.	II.	III.
Experiment 1	0.0654 gram.	0.0686 gram.	0.0654 gram.
" 2	0.0375 "		0.0377 "
" 3	0.0457 "		0.0458 "
" 4	0.0436 "		0.0437 "

It should here be noted that in regarding all the carbonaceous residue as extraneous matter, and subtracting its weight from the total weight of the precipitate, we leave all the causes of loss in our determinations unbalanced. We estimate as sulphide of antimony the material which bears a temperature of 300° unchanged, and dissolves in hydrochloric acid; and every known cause of error must tend to diminish the weight obtained.* But less sulphide of antimony corresponds to a higher apparent atomic weight of antimony; and hence, in those determinations in which the weight of the insoluble residue has been taken into account, the tendency of all the known errors must be to increase the apparent atomic weight, and, so far as our knowledge goes, it would seem impossible that the value obtained in 13 D., for example, should be too low. Moreover, the black residue is not always wholly carbon, and at times contains some antimony compound. Any siliceous or other insoluble material which accident had introduced into the analysis would of course be eliminated as a part of the residue; and the same would be true of all forms of organic matter, as well as tartaric acid, which the precipitate might absorb from the solutions in which it formed, or from the water by which it was washed.* It may be interesting, in this connection, to add a few determinations of the relative amount of combustible and incombustible matter in a few of these residues: —

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* Besides the causes of loss we have mentioned, and the small mechanical losses incident to every process of the kind, we must not overlook the fact that under most conditions a precipitate of sulphide of antimony is slightly soluble in the surrounding menstruum, and in our determinations this was frequently indicated by a barely visible coloration of the filtrate. Moreover, in several instances, we observed in this filtrate a condition which is familiar in titrations of silver; namely, a state in which either a solution of antimony or a solution of hydric sulphide would strike a red coloration.

No. 1.	Weight of residue	0.0067	grammes
	„ after ignition	0.0032	„
	Combustible	0.0035	„
No. 2.	Weight of residue	0.0078	„
	„ after ignition	0.0013	„
	Combustible portion	0.0065	„
No. 3.	Weight of residue	0.0064	„
	„ after ignition	0.0020	„
	Combustible portion	0.0044	„

Although this discussion of the causes of error was essential to the refinement of the process, it must not be inferred that the magnitude of these errors was proportional to the attention they have necessarily received, or that they are important except with reference to the accuracy required in the determination of an atomic weight. Except in those cases where, as has been stated, the amount was accurately determined and allowed for, the greatest possible error in either direction arising from occluded materials never exceeded a few thousandths of the weight estimated, and might safely have been neglected in an ordinary analysis. These errors, as we have stated, tend to balance each other; and their maximum effect is shown in the first two groups of determinations in the table on pages 36–7. In the group of five determinations 8–12, we endeavored, as has been stated, so to regulate the conditions that the opposite errors should balance each other; and the very remarkable agreement between these results and the mean of the first five is in itself a proof that we have been successful, and, moreover, their close agreement among themselves indicates that, so far as the mere mechanical details of the process are concerned, a perfection has been reached which will compare favorably with the most accurate methods of quantitative analysis. In order to exhibit the details of the work as far as practicable, we give below two examples of the most trustworthy determinations. The first is selected from the group 8–12, and is one in which no account is taken of the occluded material, but where it is believed that the conditions were so regulated that the two chief sources of error must on the average (not, however, necessarily in every determination) balance each other. This example is selected, because, although no account was taken of the carbonaceous residue, it was determined and examined, and thus some data are given for judging not only how large the errors were, but also how nearly they were balanced. The second example is unique. In this case, by a fortuitous concurrence of conditions, there was no oxichloride formed even in a solution containing an excess of hydrochloric acid, and no material sublimation of any antimony product. Here, then, we have fortunately a determination belonging to the same class as the others, in which the error is known to be all on one side, and where the error can be corrected by determining the carbonaceous residue. Since, moreover, in the details of its execution, this determination was faultless in every respect, the result it furnishes is of very great value as a standard of comparison. And, further, since the tendency of every known error not corrected by subtracting the carbonaceous residue is in the opposite direction, it seems impossible, at least, with our present knowledge, that the atomic weight of antimony should be greater than the value thus obtained.

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* In one case, when no tartaric acid was used whatever, and when the antimony was kept in solution by hydrochloric acid only, as mentioned on page 22, we obtained a distinct carbonaceous residue, evidently from some organic material which the precipitate like a mordant had absorbed.

DETAILS OF DETERMINATION MARKED No. 11 E.

The finely pulverized antimony was weighed in a platinum nacelle. By means of a loop of platinum wire, this nacelle was lowered into a glass flask having a capacity of about 150 cubic centimetres. The powder having been shaken out, the nacelle was withdrawn, replaced on the balance pan, and weighed.

Weight of nacelle and antimony . . .	8.2655	grammes.
” ” ”	6.2617	”
Weight of antimony transferred to flask .	2.0038	”

The antimony was next dissolved without the aid of heat in 30 cubic centimetres of hydrochloric acid mixed with 5 cubic centimetres of decim-nitric acid. Before complete solution had taken place, the flask was connected with a reversed condenser, and the solution boiled until every trace of the metal had disappeared. Next, the antimony bullets, previously prepared as described, were added, and the boiling continued until the solution was perfectly colorless. To this solution, after cooling, was added 20 grammes of tartaric acid, dissolved in about twice its own weight of water, and the solution transferred, in the manner before described, to an Erlenmeyer flask holding about two litres.

First weight of platinum tunnel and antimony bullets,	96.8646	grammes.
Second " " " " " " "	96.5267	"
Weight of antimony dissolved during reduction .	0.3379	"
" " " as above	2.0038	"
Total weight of antimony taken	2.3417	"

The surfaces of the bullets were not disintegrated, and there was no appearance of any antimony powder arising therefrom. After precipitation, at the ordinary temperature, the precipitate was gently digested for some time with a large excess of $H_2S + Aq$ before heating the liquid to the boiling point. Every detail was successful. The precipitate was washed six times, and collected as usual by reverse filtering in a large platinum crucible, and dried in air bath at from 120° to 130° C.

Weight of small paper disk	0.0388	grammes.
" " large platinum crucible . .	180.8345	"
	180.8733	"
Weight of crucible, filter, and Sb_2S_3 .	184.1580	"
Weight of red Sb_2S_3, dried at 130° . .	3.2847	"

A portion of the dried precipitate was dissolved in hydrochloric acid. It gave no residue. The rest was then transferred to a platinum nacelle, and heated, as has been described, in a current of pure and dry hydrogen gas.

Weight of platinum nacelle	6.2616	grammes.
" " " " and Sb_2S_3	9.0480	"
Weight of Sb_2S_3 transferred	2.7864	"

Weight after heating to 175° for half an hour	9.0433	grammes.
„ „ „ „ „ half an hour longer	9.0433	„
Loss at this temperature	0.0047	„
Loss calculated for whole amount of Sb_2S_3	0.0055	„
Weight after heating to 210°, and converting to gray modification	9.0430	grammes.
Total loss after conversion	0.0050	„
Loss calculated for whole amount of Sb_2S_3	0.0059	„

At 175° we observed a distinct sublimate, and at 210° a slight addition to it.

Weight of red Sb_2S_3, as above	3.2847	grammes.
Loss when dried at 175°	0.0055	„
Corrected weight of red Sb_2S_3	3.2792	„
Weight of antimony taken	2.3417	„
Weight of sulphur in combination	0.9375	„

0.9375 : 2.3417 = 48 : 119.90, resulting value of Sb.

Weight of red Sb_2S_3, as before	3.2847	grammes.
Loss when dried at 210°, and converted	0.0059	„
Corrected weight of gray sulphide	3.2788	„
Weight of antimony taken	2.3417	„
Weight of sulphur in combination	0.9371	„

0.9371 : 2.3417 = 48 : 119.94, the resulting value of Sb.

During the drying and conversion of the precipitate, there was a distinct empyreumatic odor, and on dissolving the gray sulphide in hydrochloric acid a considerable amount of black residue. This residue was collected on a weighed disk of paper, and examined.

Weight of carbonaceous residue calculated for whole precipitate	0.0078	grammes.
Weight of incombustible portion	0.0013	„
„ „ combustible portion	0.0065	„

Here the antimonial sublimate was but small, and the carbonaceous residue was evidently the preponderating cause of error. Hence, a result decidedly below the average. But, even if we subtract the whole of the combustible portion of this residue and leave the opposite errors uncompensated, we only raise the resulting value to 120.78.

DETAILS OF DETERMINATION MARKED 13 D.

Weight of nacelle and antimony . . .	8.2578	grammes.
„ „ „	6.2522	„
Weight of antimony transferred to flask .	2.0056	„

The metal was dissolved with 30 cubic centimetres of hydrochloric acid and 5 cubic centimetres of decim-nitric acid, as in last example. But more tartaric acid, 25 grammes, was used, and in addition 20 cubic centimetres of hydrochloric acid were added to the water in the large flask before pouring in the solution through the platinum tunnel.

First weight of platinum tunnel and antimony bullets,	95.5825	grammes.
Second „ „ „ „ „ „ „	95.2038	„
Weight of antimony dissolved during reduction . .	0.3787	
„ „ „ as above	2.0056	„
Total weight of antimony taken	2.3843	„

Here the same remarks apply as were made at the corresponding stage of the previous example.

Weight of small paper disk	0.0436	grammes.
„ „ large platinum crucible . .	180.8315	„
	180.8751	„
Weight of crucible, filter, and Sb_2S_3 .	184.2290	„
Weight of Sb_2S_3, dried at 130°	3.3539	„

Most of the precipitate was transferred to a platinum nacelle, and heated as before described.

Weight of platinum nacelle	6.2518	grammes.
„ „ „ „ and Sb_2S_3 .	9.4242	„
Weight of Sb_2S_3 transferred	3.1724	„

Weight after heating for some time at from 175° to 195°	9.4200	grammes.
Weight after conversion 210°	9.4200	„
Loss at either temperature	0.0042	„
Loss calculated for whole amount of Sb_2S_3	0.0044	„

Only a very faint sublimate was formed, but we noticed a very marked empyreumatic odor. We therefore infer that the loss of weight was caused wholly by the decomposition of the occluded tartaric acid, and on dissolving the gray sulphide in hydrochloric acid we obtained a large amount of carbonaceous residue, which was collected on a paper disk, and weighed.

Weight of paper disk, dried	0.0280	grammes.
„ „ „ „ and residue . . .	0.0400	„
„ „ residue	0.0120	„
Weight of residue calculated for whole amount of Sb_2S_3	0.0126	„

Hence we have —

Weight of Sb_2S_3 dried at 130°, as before		3.3539	grammes.
Loss on heating to 210° . . .	0.0044		
Carbonaceous residue	0.0126	0.0170	„
Estimated as pure Sb_2S_3		3.3369	„
Weight of antimony taken		2.3843	„
Weight of sulphur in combination . . .		0.9526	„

$0:9526:2.3843 = 48:120.14$, the resulting value of Sb, whether we take the red or the gray sulphide.

The above examples will illustrate how the results were obtained which are tabulated on pages 36–7.

SYNTHESIS OF

No.		Wt. in gram of Sb taken.	Wt. dissolved from balls.	Total weight of Sb used.	Wt. of red Sb_2S_3 dried at 180° C.
1.	A.	2.0036	0.2023	2.2059	3.0898
2.	E.	2.0017	0.2662	2.2679	3.1778
3.	E.	2.0113	0.0853	2.0966	2.9383
4.	A.	1.9973	0.0798	2.0771	2.9051
5.	E.	2.0019	0.1087	2.1106	2.9508
		Mean of 5 Determinations		....	....
6.	A.	1.7638	0.0430	1.8068	2.5301
7.	A.	2.0275	0.0894	2.1169	2.9639
8.	B.	2.0116	0.0188	2.0304	2.8423
9.	B.	2.0027	0.1000	2.1027	2.9429
10.	E.	2.0015	0.1424	2.1439	3.0025
11.	E.	2.0038	0.3379	2.3417	3.2792
12.	E.	2.0014	0.2168	2.2182	3.1061
		Mean of last 5 Determinations		....	....
13.	D.	2.0056	0.3787	2.3843	3.3369
		Mean of the 13 Determinations		....	....

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* Large residue of carbon, small sublimate.
† Small residue of carbon, large sublimate.
‡ Weight of black sulphide not found.
§ Both residue and sublimate small.

SULPHIDE OF ANTIMONY.

Per cent of S in same.	Corresponding At. Wt. of Sb when S=32.	Wt. of black Sb_2S_3 dried at 210° C.	Per cent of S in same.	Corresponding At. Wt. of Sb when S=32.
28.61	119.79	3.0881	28.57	120.02*
28.63	119.64	3.1764	28.60	119.82*
28.65	119.56	2.9350	28.57	120.03*
28.50	120.41	2.9021	28.43	120.85†
28.47	120.57	2.9486	28.42	120.89†
28.572	119.994	. . . .	28.518	120.322
28.59	119.91	. . . .	. . . .	. . . . ‡
28.57	119.97	. . . .	. . . .	. . . . ‡
28.57	120.04	2.8410	28.53	120.23§
28.55	120.13	2.9409	28.50	120.41§
28.58	119.94	2.9981	28.49	120.47§
28.59	119.90	3.2788	38.58	119.95§
28.59	119.92	3.1022	28.50	120.44§
28.576	119.986	. . . .	28.520	120.298
28.55	120.14	3.3369	28.51	120.14 ‖
28.5731	119.994	. . . .	28.522	120.295

‖ No sublimate or loss of weight on drying and conversion. Residue large, but weighed and subtracted. The best determination, and as perfect as can be made under conditions.

This point in our investigation was reached in the spring of 1876, and the results given above were presented to the American Academy of Arts and Sciences at their meeting of June 14th, 1876.* But although they agreed so closely with the results of Schneider, and although the close confirmation of his analysis thus furnished by our synthesis seemed so conclusive, yet we could not rest satisfied so long as the great discrepancy between this value of the atomic weight and the higher number obtained by Dumas remained unexplained. We therefore determined to repeat his experiments before publishing our results. Accordingly, in the autumn of 1876, we purified and analyzed a large number of different specimens of antimonious chloride, and the results are united in the following table. Beginning with crystallized chloride of antimony obtained from different dealers, and pure in a commercial sense, we first boiled for several hours the melted chloride over finely pulverized metallic antimony, using for the purpose a glass retort, so tilted that the condensed liquid flowed back into the body of the vessel. When boiled in this way, the surface of the vapor is marked by a very well-defined ring in the neck of the retort; and by regulating the lamp this ring can readily be maintained very near the mouth, so that, while all the chloride of antimony condenses and flows back, any more volatile admixtures will gradually escape. The retort having been brought into its normal position, the chloride of antimony was next distilled; and, rejecting the first and last eighth which came over, the rest of the product was redistilled over strips of metallic zinc, and so on three or four times, rejecting at each distillation the first and last of the product. The final distillate was then still further purified by repeated crystallizations from fusion. As the fused mass solidifies quite slowly (indicating a large loss of latent heat), it is easy to arrest the process at any point, and pour off the still liquid portion from the crystals which have formed. The last can then be remelted, and the process repeated, and so on indefinitely as long as the material lasts. In this way, from several kilogrammes of the commercial chloride we obtained the few grammes of beautifully clear and perfect crystals used in our analyses. In the fifth preparation, the crystals were obtained not by fusion, but by cooling a saturated solution of the previously distilled chloride in purified sulphide of carbon. Such a solution, saturated at the boiling point of sulphide of carbon, deposits the larger part of the chloride, when cooled to the ordinary temperature. Naturally, every precaution was taken during the course of these preparations to protect this exceedingly hygroscopic substance from contact with moist air, and all the transfers were made in a portable photographic developing chamber, the air of which was kept dry by dishes of sulphuric acid. The portions for analysis were transferred, in this chamber, to tightly fitting weighing tubes; and, after the weight was taken, they were dissolved in a concentrated aqueous solution of tartaric acid, using about 5 grammes of tartaric acid to each gramme of chloride of antimony. The solutions were then diluted, and precipitated with argentic nitrate, weighing out in each case the amount required, so that only the least possible excess of the reagent should be added. The precipitates were washed and collected by reverse filtering in platinum or porcelain crucibles, and dried in an air bath at temperatures varying from 110° to 120°. They were weighed with the small disk of paper used in this process, and the invariability of the weight of these paper disks was repeatedly tested. Also, in several instances after removing the filter, the argentic chloride was heated to incipient melting; but, as in no case was its weight thus altered, this additional precaution seemed unnecessary. In the determination numbered 17, an attempt was made to ascertain whether the presence of antimony in the tartaric acid solution appreciably influences the precipitation of argentic chloride. In this analysis, the antimony was first separated from the solution by H_2S; and, the excess of this reagent having been removed by warming the filtrate with a small amount of ferric nitrate, the chlorine was precipitated in the usual way. The result, as will be seen, agrees as nearly as could be expected with those obtained by at once precipitating the chlorine from the antimony solution; and it was not until some time subsequently that the causes of error referred to on page 5 were discovered.

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* These Proceedings, vol. xii. p. 282.

The letters in the following table indicate different preparations, thus:—

a was made from a crystallized chloride of Veron and Fontaine, Paris, which was purified in the manner described above.

b was made from a crystallized chloride marked Rousseau Frères, Paris, purified as before.

c was the same as *b*, again distilled and crystallized.

d, the same as *c*, after ten additional distillations.

e, the same as *d*, again distilled below 100° in a current of dry hydrogen gas.

f was made with a crystallized chloride from Merck of Darmstadt, purified by repeated distillations and subsequent crystallizations from bisulphide of carbon, after first treating with chlorine as described beyond.

g, same as *f*, but in this determination the antimony was first precipitated from the solution.

Analysis of Antimonious Chloride.

Determination of Chlorine.

No.	$SbCl_3$. grammes.		AgCl. grammes.	% of Chlorine. Cl = 35.5. Ag = 108.	At. Wt. of Sb. Cl = 35.5
1 *a.*	1.5974	yielded	3.0124	46.653	121.78
2 *a.*	1.2533	,,	2.3620	46.623	121.93
3 *a.*	0.8876	,,	1.6754	46.696	121.57
4 *b.*	0.8336	,,	1.5674	46.516	122.46
5 *b.*	0.5326	,,	1.0021	46.546	122.30
6 *b.*	0.7270	,,	1.3691	46.588	122.10
7 *c.*	1.2679	,,	2.3883	46.599	122.04
8 *c.*	1.9422	,,	3.6646	46.678	121.66
9 *c.*	1.7702	,,	3.3384	46.655	121.77
10 *d.*	2.5030	,,	4.7184	46.635	121.87
11 *d.*	2.1450	,,	4.0410	46.616	121.96
12 *e.*	1.7697	,,	3.3281	46.524	122.42
13 *e.*	2.3435	,,	4.4157	46.613	121.98
14 *f.*	1.3686	,,	2.5813	46.659	121.75
15 *f.*	1.8638	,,	3.5146	46.650	121.79
16 *f.*	2.0300	,,	3.8282	46.653	121.78
17 *g.*	2.4450	,,	4.6086	46.630	121.89
	Mean value for all analyses			46.620	121.94
	Theory when		Sb = 122	46.608	122.
	,,	,,	Sb = 120	47.020	120.

If in calculating the per cent of chlorine from the results of the above determinations we use the atomic weights for silver and chlorine obtained by Stas (namely, Cl = 35.457 and Ag = 107.93), these per cents will be in each case very nearly 0.020 lower, and we shall obtain for the mean value 46.600 instead of 46.620. Moreover, on this assumption the atomic weight of antimony, deduced from Dumas's analysis of the chloride, would be 121.95 instead of 122. Again, if we use Stas's value of the atomic weight of sulphur (S = 32.074) in calculating the atomic weight of antimony from our own results, on the synthesis of the sulphide, we should obtain 120.28 instead of 120; and, lastly, the values Sb = 120.28 and Cl = 35.457 give for the per cent of chlorine in antimonious chloride the value 46.931.

Here, then, is a most striking result; for these determinations confirm the value of the atomic weight of antimony obtained by Dumas as closely as did the previous determinations confirm that obtained by Schneider. Evidently, there was a large constant error in one case or the other. Moreover, it appeared improbable that in either case any error could arise in the chemical process employed: for, in the first instance, we had a synthesis by one method confirming an analysis by a wholly different method; and, in the second instance, the analytical process employed is regarded as one of the most accurate known to science, and we had apparently shown that its accuracy was not impaired under the peculiar conditions present. It appeared, therefore, reasonable to assume that the results did truly indicate both the actual proportion of antimony in the sulphide of antimony and of chlorine in the chloride of antimony analyzed, and to look for the cause of the discrepancy to some impurity in one or the other compound. We therefore next sought to determine how much sulphide of antimony could be obtained from a given weight of chloride of antimony, hoping that by thus bringing the relations of antimony to chlorine and sulphur into close comparison the source of the error might be indicated.

The precipitation of sulphide of antimony from a solution of the chloride in tartaric acid is made under conditions which are far more favorable to accurate results than those with which we were obliged to contend in our previous experiments. If the solution is diluted with water charged with carbonic dioxide, and if the precipitation is made in the way we have described,* we can avoid all excess of hydrochloric acid except that which is formed by the chemical reaction, and there is then no oxichloride of antimony formed, and no subsequent loss of chloride of antimony on heating the red sulphide up to the point of its conversion to the gray modification. Moreover, less tartaric acid is required, — not more than 6 grammes to every gramme of chloride, — and consequently less is occluded by the precipitate, so that with care the carbonaceous residue can be reduced to an insignificant amount.

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* See page 14.

The following table exhibits the results of these antimony determinations, as well as the general result of the assumed complete analysis of antimonious chloride. The per cent of chlorine taken is the mean of the first thirteen determinations of the previous table, as these only had been made at the time the second table was drawn up, and it therefore exhibits the results exactly as they were presented to us at this stage of the investigation.

ANALYSIS OF ANTIMONIOUS CHLORIDE.

DETERMINATION OF ANTIMONY.

	$SbCl_3$ taken in grammes.	Sb_2S_3 obtained in grammes.	% of Antimony when Sb : S = 120 : 32.*	% of Antimony if Sb : S = 122 : 32.†
1 *b.*	3.8846	2.8973	53.275	53.525
2 *b.*	5.1317	3.8417	53.473	53.725
3 *b.*	4.4480	3.3201	53.316	53.567
4 *b.*	4.5506	3.4009	53.882	53.633
5 *b.*	4.8077	3.6072	53.593	53.845
6 *b.*	4.2774	3.1958	53.367	53.618
Mean of all Analyses			53.401	53.652

MEAN RESULT OF COMPLETE ANALYSIS.

Antimony, the mean of six determinations	53.401	53.652
Chlorine, „ „ thirteen „	46.611 ‡	46.611 ‡
	100.012	100.263

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* Or assuming that $\frac{5}{7}$ of the gray sulphide is antimony, as deduced from actual synthesis.

† According to the generally accepted theory.

‡ When Cl = 35.5 and Ag = 108, according to Dumas.

As they at first presented themselves to us, these new results, so far from throwing light on the subject, only rendered the problem the more obscure and baffling. Towards interpreting them however, one point seemed evident; — that, however little value our own experiments and those of Schneider might have in fixing the atomic weight of antimony, they had at least established, beyond all doubt, the proportion of this element in the gray sulphide weighed in our antimony determinations. For if we assumed, as those experiments indicated, that five-sevenths of the gray sulphide was antimony, then the amounts of antimony and chlorine found in the analysis of antimonious chloride just made almost exactly supplemented each other; while on the other hand, if this material was, as generally believed, pure Sb_2S_3, in which $Sb : S = 122 : 32$, then our determinations of one or the other of these elements must be greatly erroneous, and the excess obtained far too great to be explained by any known or probable imperfections of our methods. Of course, although the gray sulphide might contain, on the average, five-sevenths of its weight of antimony, it was a possible supposition that it might also occlude a constant amount of some undiscovered impurity, leaving the proportion of the sulphur to the antimony that which the atomic weights 122 and 32 required; and, were it not for our previous experience, this would have been the most obvious explanation of the discrepancy. Indeed, the new facts led us to re-examine this material, and review our previous conclusions, but with the same result as before. We could discover no impurity except the small amount of carbonaceous material which was so well known and taken into the account, and in our later determination (as is shown by the example on page 69) even this had been reduced to so small an amount as to be wholly insignificant. It is very difficult in any such case to prove a negative, but in the present instance the following considerations appeared conclusive, which on account of their important bearing we in great part recapitulate, although the evidence was again reviewed at this time.

1. Such an impurity must be comparatively large in quantity, not less than several centigrammes in the amount of the gray sulphide usually weighed.

2. It must exist to an equal extent in both the native sulphide and the artificial product of our analytical process, for both contain the same percentage of antimony.

3. It must be derived from antimonious chloride, tartaric acid, hydric sulphide, and water, which were the only reagents used in the process, and from these in the purest condition they could be obtained.

4. It must be able to bear a temperature of 300° C. without alteration, for in our later determinations the sulphide was heated to this temperature.

5. It must be soluble in hydrochloric acid; for allowance was made for the impurity which remained undissolved when the sulphide was decomposed by this reagent.

Hence, it still appeared, as before, quite impossible that any such impurity could be present; but, in order to eliminate even more certainly any unforeseen and accidental conditions, we made the following experiments.

In the first place, we brought together the aqueous solutions of hydrochloric acid, tartaric acid, and sulphide of hydrogen in the same proportions, and under the same conditions in which they were used in our antimony determinations; but, although we made several trials, we obtained in no case the slightest precipitate.

In the second place, we heated a small amount of the tartaric acid used to 300° C., treating it exactly in the same way as the precipitated antimonious sulphide in our analytical processes; and we found that the small amount of carbonaceous residue obtained was wholly insoluble in hydrochloric acid.

It would evidently have been more satisfactory to be able to determine the amount of combined sulphur in the gray sulphide, and thus to prove that it exactly supplemented the antimony, as we had endeavored to do in the case of the antimonious chloride; but unfortunately we could devise no method which promised to be satisfactory. We were obliged therefore to rest on the negative evidence, and this seemed to exclude the most obvious explanation of our contradictory results. Of course, we could not reasonably entertain the question that the large excess which the analyses of table pages 40 and 42 presented, when calculated for $Sb = 122$, was due wholly to an overestimate of the chlorine, because the amount of chlorine was only that which this theory of the composition of $SbCl_3$ required; and, moreover, the constancy in the composition of the chloride, after it had been submitted to such various treatment, was surprisingly great, and served to exclude the idea of any error at all comparable with that which the excess in question would imply. We had found apparently that the composition remained sensibly constant, even after the successive distillation of what we had regarded as essentially pure material. We had distilled it in a current of perfectly dry hydrogen, at a temperature below 100°, and still obtained in the distillate the same composition as before. We now further made the four additional chlorine determinations, Nos. 14 to 17, in the table on page 40. The material for these analyses was first purified by repeated distillations as before, first over powdered antimony and then over zinc. Through the melted chloride was then passed, for a short time, a current of dry chlorine gas, in order to make sure that no such thing as a subchloride of antimony, if it can exist, or particles of metallic antimony, could be present. The chloride was then again redistilled several times, and this product purified by ten or twelve repeated crystallizations from a solution in disulphide of carbon, the material being protected all the time as far as possible from contact with moist air. The few grammes of pure chloride thus obtained from more than a kilogramme of so-called pure commercial chloride of antimony were submitted to analysis, and, as will be noticed, the results were in complete accordance with those we had obtained before. Lastly, we found nearly the same result, with only the little loss that was to be expected, when the antimony was removed from the solution before precipitating the chlorine.

Now, it is evident that, if the sulphide of antimony we weighed is pure, we are forced, even by these last analyses, to the conclusion that the atomic weight of antimony must be very nearly at least 120, if that of sulphur is 32, although the singular discrepancy which our results presented served at the time to render the problem exceedingly puzzling. The facts indeed seemed to indicate that, while antimony combined with chlorine in the proportion of 122 to 3 times 35.5, it combined with sulphur in the proportion of 120 to $\frac{3}{2}$ times 32, or 122 to $\frac{3}{2}$ times 32.53; in other words, that the relation of the atomic weights of chlorine and sulphur was not as accepted 35.5 to 32, but 35.5 to 32.53. And, although, after the investigations of Dumas, Stas, Marignac, and others, it was clearly out of the question that these values should be in error to the extent indicated, yet, as we have seen, Stas had found for the atomic weight of sulphur 32.074, and the results of our synthesis of sulphide of antimony calculated on this basis would give for the atomic weight of antimony 120.28. Moreover, it appeared that when the analyses of antimonious chloride made by Dumas were recalculated with Stas's values of the atomic weights of chlorine and silver ($Cl = 35.457$, $Ag = 107.93$), they gave for the same atomic weight the number 121.95. This reduced the difference between the two determinations to 1.67, and it did not seem impossible that the whole discrepancy might result from the accumulation of a number of similar small errors. We were thus led to undertake a new comparison of the atomic weights of chlorine and sulphur, based on the precipitation of sulphide of silver, by the same process we had employed in precipitating sulphide of antimony, not with any expectation of correcting the atomic weight of sulphur, but with a view possibly of verifying the higher value obtained by Stas, and more especially of still further testing the accuracy of our method of precipitating sulphides; as it was obvious that any hidden sources of error which could have impaired the accuracy of our antimony determinations might be expected to reappear in the experiments with silver, and then the well-established composition of sulphide of silver would help us to detect them.

We began these experiments by taking two adjacent portions of the same piece of pure silver foil, and, having dissolved each in nitric acid, we in the first place precipitated the silver from both solutions, as chloride, with the usual precautions. The argentic chloride from the first solution was washed, collected, and weighed as before described. That from the second solution, having been thoroughly washed by the process of reverse filtering, was redissolved in the same vessel with pure aqua-ammonia, and from this solution the silver was precipitated as sulphide by adding the supersaturated solution of hydric sulphide, using the same precautions with which we were familiar in the precipitation of sulphide of antimony. The sulphide became granular on boiling, and was readily washed and collected by the method of reverse filtering. We thus hoped to obtain a direct comparison of the atomic weights of chlorine and sulphur, not depending on the absolute purity of the metallic silver used, and moreover to obtain a confirmation or otherwise of the general accuracy of our method of determining sulphide of antimony; for it was obvious that the same causes of error were likely to inhere in two such similar processes. Hence, although such experiment would probably only confirm values already well established, such a confirmation would give us confidence in the accuracy of our previous work. But although the mechanical details of the process appeared perfect, and the results were not inconsistent with the accepted values of the weights under discussion, yet they were neither sufficiently sharp nor constant to answer the questions we had proposed, owing probably to some slight solvent action of the ammoniacal menstruum on the precipitated sulphide of silver. We were therefore led to modify the process by first preparing pure sulphide of silver by the method we have described, and then determining the relation of silver to sulphur by reducing weighed portions of this sulphide in a stream of hydrogen gas. This result, compared with the already well-known relation of silver to chlorine, — probably the most accurately determined of all the atomic ratios, — would give us the relation of sulphur to chlorine which we sought, and under essentially the same conditions as before. As thus modified, the method gave exceedingly sharp results, and the whole course of the analytical process seemed favorable to extreme accuracy. The reduction of the sulphide takes place at a temperature far below the melting point of silver, — indeed, below a visible red-heat, — and the metal separates in a most beautiful fine filamentary condition, which very greatly facilitates the reducing action of the hydrogen gas. Here again, however, we encountered another of those unforeseen constant errors which have caused us so much perplexity and fruitless labor during the whole investigation; and we give in the following table a series of results which have no other value than as illustrating the remark we have before made, that no amount of accordance in the results of the same analytical process is a sufficient guarantee against errors of this class: —

REDUCTION OF ARGENTIC SULPHIDE.

FIRST SERIES OF EXPERIMENTS AT FULL RED HEAT.

No.	Weight of Ag_2S, taken in grammes.	Weight of S, or loss during reduction.	Corresponding At. Wt. of S when Ag = 108.
1, *a.*	1.3380	0.1749	32.48
2, *b.*	1.2089	0.1582	32.52
3, *c.*	2.6592	0.3476	32.48
4, *c.*	1.9419	0.2540	32.50
5, *d.*	3.2784	0.4283	32.47
6, *e.*	2.4036	0.3143	32.49
7, *f.*	2.8359	0.3711	32.52 *

Mean value	32.494
Extreme variation from mean	.026

The sulphide used in the first five determinations was prepared from washed chloride of silver, which was dissolved in pure aqua-ammonia and precipitated with $H_2S + Aq.$, as described above. The portions used in the last two determinations marked *e* and *f* were precipitated from ammonio-argentic nitrate, prepared in the first case from metallic silver, and in the last case from crystallized nitrate of silver, with the least possible excess of ammonia. During the process, a current of CO_2 was passed through the solution, which was boiled with the precipitate until all odor of ammonia had disappeared, when violent bumping ensued. After having been collected and dried, the sulphide was heated, in the first case to 280°, and in the last case to 300°, in a current of CO_2. We observed no sublimate, and there was no loss of weight with either preparation. Of the preparation *d*, seven grammes were boiled with strong hydrochloric acid until completely converted into chloride, which was then tested for sulphur, but none could be detected. The products of the decomposition of several of the preparations were tested for ammonia, and other volatile bases which might possibly have been occluded by the precipitated sulphide. Ammonia was at first found in abundance, but this was soon traced to an impurity in the hydrogen used in the reduction; and after this source of error was removed not the smallest quantity of any such product could be detected. It will thus be seen that these very closely according results were obtained with seven different preparations, made in part by different processes; and, further, that we had the strongest evidence of the purity of the material used; and, lastly, that the perfect metathesis of the sulphide with hydrochloric acid, leaving no residue of either elementary substances, proved that the silver was united to the sulphur in atomic proportions. Nevertheless, all these determinations were in error, and the error arose in this way: —

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* In this determination, the reduced silver fused.

The argentic sulphide, held by a porcelain nacelle in a porcelain tube heated by a gas furnace, was reduced in a current of hydrogen gas. The hydrogen was prepared in an automatic generator, from clippings of sheet zinc, common sulphuric acid, and water, in the usual way; but, before reaching the porcelain tube, passed through a very long series of purifiers and driers, containing in order quick lime, soda lye, solution of acetate of lead, solution of argentic nitrate, sulphuric acid, and chloride of calcium. The object we had chiefly in view was to remove from the gas any traces of arseniuretted hydrogen or similar compounds; and we felt assured of its purity in this respect, on finding that it formed no metallic mirror even after passing for several hours through a glass tube heated to redness, and caused no darkening of "lead paper" after very prolonged exposure. The reduction of the sulphide of silver was of course attended with the evolution of hydric sulphide, and the process was continued until no trace of this substance could be detected with "lead paper" in the gas which escaped from the porcelain tube. It was evident from the first that, as already stated, the reduction could be at least nearly completed below visible redness; and, as we now know, argentic sulphide can be perfectly reduced at this low temperature, but in all the earlier determinations we found it necessary to raise the temperature at the end of the process to a full red heat, and continue the heating for several hours before the untarnished lead paper indicated that the evolution of hydric sulphide had ceased. We subsequently discovered that this effect was due to the presence of the finely divided silver, determining a reducing action of the hydrogen on certain impurities which the gas had contracted from the crude sulphuric acid, and which had escaped all the purifiers. These impurities contained both nitrogen and sulphur, and, when reduced by hydrogen in presence of the metallic silver, yielded both ammonia — to which we have already referred — and also the hydric sulphide which had misled us in regard to the completion of the reduction. On passing the gas through a glass tube containing platinum sponge, heated to low redness, the effect was still more marked; and, on placing this tube after the first alkaline purifier, the products just named were evolved in abundance, although previously, even at this point, the gas produced no effect on "lead paper." We thus traced the impurity back to the hydrogen generator, and were able to remove it by placing in the line of the purifiers a glass tube filled with platinum sponge, and heated by a combustion furnace. The tube was placed, as above described, after the alkaline purifier; and from this the gas was passed through several purifiers containing a solution of nitrate of silver, and through driers containing in part sodic hydrate and in part calcic chloride. Afterwards, the nitrate of silver in the purifiers was replaced by alkaline and acid solutions of potassic permanganate, as recommended by Schobig,* which were, at least, equally efficient. The crude acid used in the generator was found to contain an unusual amount of nitric acid, and the impurity contracted by the hydrogen was probably some volatile compound of oxygen nitrogen and sulphur, similar to that with which we are so familiar in the sulphuric acid chambers. Misled, as we have seen, by the indications of sulphur, we continued the reduction — in the experiments whose results have been given on page 47, — far longer and at a far higher temperature than was necessary; and the apparent increased value of the atomic weight obtained was due to a slight volatilization of the metallic silver. Except in one of the experiments, the temperature had never reached the melting point of silver; but, as soon as our suspicions were aroused, we detected a slight mirror near the open end of the reduction tube, which, when dissolved off with a few drops of nitric acid, and tested with hydrochloric acid, gave abundant evidence of what had taken place. And, by subsequent experiments in a stream of hydrogen gas, we found that, under the conditions present in our experiments, silver volatilized — very slightly, it is true, but markedly — at a temperature considerably below its melting point.

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* Jour. pr. Ch. (2), xiv. 289, Oct., 1876.

We have thus shown, first, that recently reduced silver exerts a catalytic action precisely similar to that of spongy platinum, although not so powerful; and, secondly, that under these conditions the silver slowly volatilizes, at a temperature considerably below its melting point. Whether the volatilization is increased by the catalytic action, or why, in our experiments, the loss should have been so constant in amount, we have had no opportunity to determine. It was only necessary at the time to establish the fact that the results were vitiated by this constant error, and we at once hastened to determine whether the sulphide could be perfectly reduced at a temperature below that at which silver volatilizes in a current of pure hydrogen. To this end, we made many experiments, carefully testing in each case the reduced silver for sulphur, and examining with the greatest care the interior surface of the porcelain tube for any evidence of volatilization. We thus found that, by regulating the temperature, sulphide of silver could be perfectly reduced at a low red heat, without giving any evidence of loss from this cause. The following determinations were made in this way: —

Reduction of Argentic Sulphide.

Second Series of Experiments at Low Red Heat.

No.	Wt. of Ag_2S, grammes.	Wt. of S by loss.	At. Wt. of S. $Ag = 108$.	At. Wt. of S. $Ag = 107.93$.
1, *a.*	7.5411	0.9773	32.160	32.139
2, *a.*	5.0364	0.6524	32.143	32.122
3, *b.*	2.5815	0.3345	32.155	32.134
4, *c.*	2.6130	0.3387	32.168	32.147
5, *d.*	2.5724	0.3334	32.164	32.143
Mean value			32.158	32.137
Extreme variation from mean .			0.015	0.015

In this series, as in the last, the letters indicate different preparations. The two marked *a* and *b* were both made from washed chloride of silver, dissolved in pure aqua-ammonia; that marked *c* was made from pure nitrate of silver, first converted into ammonio-nitrate, with the least possible excess of ammonia; that marked *d* was precipitated directly from a dilute aqueous solution of the same argentic nitrate, without ammonia, and was therefore formed in an acid solution. They were all precipitated with a supersaturated solution of hydric sulphide, and during the precipitation and subsequent boiling a current of carbonic dioxide was passed through the liquid. After the material had been placed in the nacelle for reduction, it was heated to 300°, in a current of carbonic dioxide, before the weight was taken. These facts are stated, because, as will be seen, the close accordance of the results obtained furnishes the strongest evidence of the uniform purity of the material prepared in the several ways we have described, and gave us great confidence in the perfection of our new method of precipitating sulphides.

Stas obtained for the atomic weight of sulphur when $Ag = 107.93$ the value 32.074, and the mean of our results differs from his by only 0.063. How small this difference really is, is shown by the fact that even with the largest quantity of sulphide used, — which required a platinum nacelle 5 inches long by $1\frac{1}{8}$ inches wide to hold the spongiform* mass of reduced silver, — the difference in question only corresponds to $1\frac{7}{10}$ milligrammes in the weight estimated; and with the smaller quantities — which required the largest porcelain nacelle we could obtain — the difference only corresponds to about half a milligramme. Still, the process is sufficiently accurate to show even this difference; for the extreme variations from the mean value in the last series of results only corresponds for the larger quantities to $\frac{4}{10}$ of a milligramme, and for the smaller to $\frac{15}{100}$ of a milligramme of the quantity estimated. The difference, therefore, small as it is, evidently points to a constant error of some kind, which, as we suspect, is caused by a slight volatilization of silver, even at this comparatively low temperature, although we were unable to obtain any other evidence of it. Hence, the following two additional determinations may be of interest, in which the sulphide was reduced below a visible red heat, in a small platinum nacelle, heated in a tube of hard glass: —

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* The production of moss silver in this process is a most beautiful phenomenon, which has been described by Dr. Percy, "Metallurgy," I. 360, and more recently by Professor Liversidge, of the University of Sydney.

REDUCTION OF ARGENTIC SULPHIDE.

THIRD SERIES OF EXPERIMENTS, TEMP. BELOW VISIBLE REDNESS.

No.	Wt. of Ag_2S. grammes.	Wt. of S. by loss.	At. Wt. of S. Ag = 108.	At. Wt. of S. Ag = 107.93.
1.	1.1357	0.1465	31.990	31.969
2.	1.2936	0.1670	32.010	31.990
	Mean value		32.000	31.980

We are sure that in these experiments no silver was lost, because the least trace of sublimate would have been visible on the glass. We cannot be so certain that a trace of sulphide did not remain unreduced; but we do feel confident that the true value of the atomic weight of sulphur — so far, at least, as it can be determined by the analysis of argentic sulphide — must lie between the limits which the two last series of experiments fix. This is equivalent to confirming the accepted value of this constant, so far as any experiments on a scale less extensive than those of Stas can be of value to this end.

While, therefore, this portion of our investigation was not wanting in interesting results, it did not help us to explain the discrepancy we had observed in our experiments on the atomic weight of antimony. We now felt, however, greater confidence in our synthesis of sulphide of antimony; for if the sulphide of silver we had analyzed was so pure there was every reason to believe that the sulphide of antimony prepared in the same way was equally pure, save only the small occlusions which were so well known, and had been taken into account. We were therefore now still more fully persuaded that the value 120, which we had obtained for the atomic weight of antimony, must be correct within a few tenths of a unit; and it seemed to us very clear that the constant error, which had so perplexed us, was to be looked for in the analyses of chloride of antimony. Moreover, it seemed probable to us at this time that we might obtain a clew to the hidden source of error by analyzing the bromide and iodide of antimony before continuing our experiments on the chloride, for the same influences would be likely to affect all these processes; yet it was reasonable to expect that they would act in varying degrees in the three cases, and that they might thus reveal their nature. We begin with our work on the bromide.

We prepared the bromide of antimony by adding in small portions at a time the pulverized metal to a strong solution of bromine in sulphide of carbon. The retort containing the solution was kept cool by snow, and shaken after each addition until the action ceased. As soon as the color of bromine was discharged, the sulphide of carbon was distilled off over a water bath; and then, replacing the water bath with a gas lamp, the bromide of antimony was first boiled, and then distilled over the finely powdered antimony which had been added in excess. On account of the high boiling point of bromide of antimony, and the readiness with which its vapor condenses, it was found best in distilling to cover the body of the retort with a hood. The bromide thus prepared was purified by repeated distillations over pulverized antimony, as in the case of the chloride, and finally by crystallizing and recrystallizing several times from solution in purified sulphide of carbon. A warm saturated solution in sulphide of carbon deposits, when cooled to the freezing point, the greater part of the bromide of antimony in fine acicular crystals. These crystals were dried first with blotting-paper, and then *in vacuo* over sulphuric acid. The antimonious bromide thus purified by fractional distillation and crystallization was only a very small fraction of the first crude product. It was pure white, had a high silky lustre, and, when first made, was wholly destitute of odor. It was carefully examined for chlorine, iodine, and arsenic; but the delicate test which we possess for all three of these elements, so frequently associated with commercial antimony and bromine, failed to show the least trace of either in the bromide of antimony we analyzed. The determinations of bromine were made in all respects like those of chlorine. Great care was taken not to add more than a very slight excess of argentic nitrate, and we found that under these conditions the supernatant liquid cleared more readily above the precipitate in the case of bromide of silver than with the corresponding chloride, and for this reason the first could be washed more quickly than the last. The results of these determinations are embodied in the table on the following page.

Here, as before, the letters indicate different preparations: *a* was made and purified as described above; *b* was the same material as *a* redistilled and again crystallized from bisulphide of carbon; *c* was another portion of the same material several times redistilled and twice recrystallized from the same solvent; *d* was a separate preparation from the start; *e* was another separate preparation purified with extreme care. In the last case there was over a kilogramme of the crude product, which was reduced by the fractional distillation and crystallization — each process repeated from ten to twenty times — to the few grammes used in the analyses. These methods of purifying the substance were thus pushed to their utmost limits.

ANALYSES OF ANTIMONIOUS BROMIDE.

DETERMINATION OF BROMINE.

No.	Wt. of $SbBr_3$ taken in grammes.	Wt. of Ag Br obtained.	% of Bromine. Br = 80, Ag 108.
1, *a.*	1.8621	2.9216	66.765
2, *a.*	0.9856	1.5422	66.584
3, *b.*	1.8650	2.9268	66.779
4, *b.*	1.5330	2.4030	66.703
5, *b.*	1.3689	2.1445	66.663
6, *c.*	1.2124	1.8991	66.655
7, *c.*	0.9417	1.4749	66.647
8, *d.*	2.5404	3.9755	66.593
9, *d.*	1.5269	2.3905	66.623
10, *e.*	1.8604	2.9180	66.743
11, *e.*	1.7298	2.7083	66.624
12, *e.*	3.2838	5.1398	66.604
13, *e.*	2.3589	3.6959	66.671
14, *e.*	1.3323	2.0863	66.635
15, *e.*	2.6974	4.2285	66.708

Mean value from last six determinations . 66.664

Mean value from all the determinations . 66.6665

Theory Sb 120 requires 66.6666

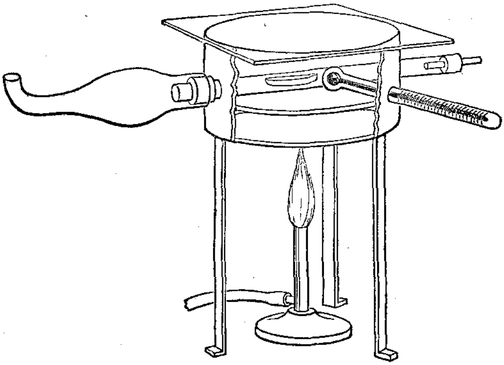
Theory Sb 122 „ 66.2983

If in calculating the results of the above bromine determinations we use the atomic weights of Stas, — Br = 79.952, Ag = 107.93, — the per cents found will be in each case only 0.002 higher, which is, of course, an inappreciable difference. Hence, whether we take Stas's or Dumas's values for the atomic weights of bromine and silver, the atomic weight of antimony deduced from the above determinations is exactly 120.00.

This is certainly a remarkably close confirmation of our previous conclusion. Indeed the wonderful coincidence between the observed and the theoretical results must be to a certain extent accidental; for no process of chemical analysis is capable of the accuracy which this agreement would imply. Still it should be noticed that the probable errors of the process, so far as they are indicated by the variations from the mean value, are not larger than we might expect would be eliminated by multiplying observations; and, further, that the mean of the last six determinations which are undoubtedly the most trustworthy, is nearly as close to the theory as the mean of the whole.

But not only did these experiments on bromide of antimony thus confirm our previous conclusion: they also gave the first definite clew to the explanation of the disagreement with otherwise consistent results which our experiments on chloride of antimony had presented. The one difference between the chloride and the bromide, which appeared to render the last better suited to yield accurate results, was the difference in their hygroscopic qualities. As we have stated, the chloride is one of the most hygroscopic substances known. The bromide is also hygroscopic, but far less so, presenting no unusual difficulties of manipulation; and, since our tests indicated that both substances were otherwise pure, we at once drew the inference that the different results we had obtained with chloride of antimony must depend on the extraordinary attraction of this substance for moisture. Before, however, fully following out the clew thus obtained, we made a similar study of the iodide of antimony.

The iodide of antimony was prepared like the bromide, by shaking up in a glass flask a solution of iodine in bisulphide of carbon with finely pulverized metallic antimony. On filtering and decanting, after the color of the iodine is discharged, a solution having a pale greenish-yellow color is obtained, from which on cooling or on evaporation red crystals of iodide of antimony are deposited. The substance may be purified by recrystallization from the same solvent; but iodide of antimony is far less soluble in bisulphide of carbon than the chloride or bromide, and cannot therefore be so advantageously treated in this way, nor can the small amount of carbonaceous impurity which the crystals acquire from the solvent be so easily removed. Moreover, iodide of antimony cannot be so readily distilled as the chloride or bromide, on account of its high boiling point, which is above that of metallic mercury. But another property of iodide of antimony which, so far as we know, has not hitherto been noticed, interferes still more seriously with these methods of purifying this substance. In all its conditions, it undergoes a more or less rapid oxidation in contact with atmospheric air, forming oxi-iodide of antimony (SbOI) and free iodine. When iodide of antimony is rapidly boiled in a small flask, so that the body and most of the neck are kept full of vapor at the boiling-point, the action at the surface of contact of the vapor and the air is very striking; iodine is set free in vapor, with its familiar violet color, while the oxi-iodide is precipitated in clouds, forming a most beautiful phenomenon. So also when the greenish-yellow solution (above described) of the iodide in bisulphide of carbon is exposed to the air and light, it rapidly becomes colored red from the liberation of iodine, and at the same time turbid from the deposition of the insoluble oxi-iodide. Even the crystals of iodide of antimony, when kept in the light, slowly become opaque from the formation of the same oxi-iodide; while the odor and staining of the stopper of the bottle furnish abundant proof of the liberation of iodine. The study of these phenomena was most interesting, and the results obtained will be described in another paper. It is sufficient for the present to say that they pointed out to us a great source of impurity in iodide of antimony, and fully explained the want of accordance in our analyses of the crystals of this substance as first prepared. It was evident that we could only hope to purify the material by distilling or subliming it in an atmosphere of inert gas; and we devised the apparatus represented in the accompanying figure for this purpose, which we have since found very generally useful for all sublimations where the temperature required does not exceed that which can be measured with a mercury thermometer. The apparatus has been already referred to (page 26), and requires no further description. It was a simple modification of the apparatus used before for drying at a regulated temperature the precipitates of sulphide of antimony, which, as we have stated, was so arranged that the character of any sublimates which might be given off could be observed. We used the same glass tube passing through the sheet-iron air-bath, with its transparent mica cover, only we added a common glass adapter, selected so that its mouth just fitted over the open end of the tube. A platinum nacelle containing iodide of antimony, which had already been purified by crystallization, was placed in the tube within the air-bath, but near the open mouth; and, while a current of dry carbonic dioxide through the apparatus was steadily maintained, the air-bath was heated by a gas lamp to the required temperature which was



indicated by a thermometer, as shown in our figure. Iodide of antimony is sensibly volatile, even at 100° ; and long before it reaches its melting point, 167°, the evaporation becomes very marked. As soon as melted, it sublimes quite rapidly ; and we obtained the best results by keeping the temperature between 180° and 200°, and, by shifting the adapters we used as receivers, it was easy to collect the different portions of the sublimate. We thus obtained crystals of two isomeric modifications of iodide of antimony : the more abundant in large hexagonal plates, often half an inch or more in diameter, perfectly transparent, and of the most brilliant ruby-red color ; the other in small rhombic plates, having the same peculiar greenish-yellow color as the solution of the iodide already mentioned. The amount of the last was always small, but it was larger in proportion as the temperature was lower. This new and most interesting product will be described in the paper just referred to. Of these crystals, the most brilliant, chiefly of the red variety, were selected for analysis. The iodine determinations were conducted in all respects like those of chlorine and bromine. The iodide was first dissolved by a very concentrated solution of tartaric acid, and then the solution was diluted to the required extent. The same care was taken not to add more than a very slight excess of argentic nitrate, and the amount required was accurately weighed out in each case. Each of the determinations was made with a separate preparation in so far as it was a product of a separate sublimation; but the material sublimed was essentially the same in all cases, — a mixture of the products of many crystallizations from the crude material made as described above. The results are collected in the following table : —

ANALYSIS OF IODIDE OF ANTIMONY.

IODINE DETERMINATIONS.

No.	Wt. of SbI_3, grammes.	Wt. of AgI, grammes.	% of Iodine, I = 127, Ag = 108.	Variety.
1.	1.1877	1.6727	76.110	Pure red.
2.	0.4610	0.6497	76.161	Chiefly yellow.
3.	3.2527	4.5716	75.956	Pure red.
4.	1.8068	2.5389	75.939	Pure red.
5.	1.5970	2.2456	75.990	Red and yellow.
6.	2.3201	3.2645	76.040	Pure red.
7.	0.3496	0.4927	76.161	Chiefly yellow.

Mean value	76.051
Theory Sb = 120, requires . .	76.047
Theory Sb = 122, „ . .	75.744

If in calculating the results of these iodine determinations we use the atomic weights of Stas, I = 126.85 and Ag = 107.93, the mean value would be 76.034, and the corresponding atomic weight of antimony 119.95.

The difference (0.004) between the first mean value and theory — corresponding to only about $\frac{2}{10}$ of a milligramme in the largest amount of argentic iodide weighed — is evidently insignificant, so that these results confirm the lower value of the atomic weight of antimony as closely as did the analyses of the bromide.*

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* After the success we had in the application of our method of sublimation to purifying the iodide, we attempted to purify the bromide of antimony in the same way. We thus obtained a very beautiful product, free from every trace of impurity except hygroscopic moisture. The last could not be avoided without more efficient means than we then had of drying the necessarily somewhat rapid gas current; but we were satisfied that with proper precautions, this would be a better method of preparing pure antimonious bromide than the one

As we have already intimated, our analyses of the iodide of antimony, as first crystallized from bisulphide of carbon, yielded very discordant results. These we give in the table below, not, as before, in the exact order in which the analyses were made, but in the order of the several values, so as to exhibit the distribution of the errors.

ANALYSES OF CRYSTALLIZED ANTIMONIOUS IODIDE, RED VARIETY.

No.	% of Iodine.
1.	75.71
2.	75.76
3.	75.78
4.	75.80
5.	75.84
6.	75.85
7.	75.87
8.	75.89
9.	75.94
Mean value	75.83
Theory Sb = 122	75.74
Theory Sb = 120	76.05

The cause of this discordance we attributed, as we have intimated, chiefly to the remarkable readiness with which iodide of antimony undergoes oxidation in contact with the air, resulting in the formation of oxi-iodide of antimony and free iodine, thus:—

$$2\,SbI_3 + O\text{-}O = 2\,SbOI + 2\,I\text{-}I.$$

While the free iodine escapes, the oxi-iodide remains as an impurity in the preparation, and the effect is a replacement of a portion of its iodine by oxygen. Now, since eight parts of oxygen replace one hundred and twenty-seven parts of iodine, it can readily be seen that an otherwise almost imperceptible amount of oxidation would be sufficient

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we employed. For the reasons stated the results of the analyses of the preparations we made in this way were not as concordant as those exhibited on page 54, although the close agreement of the mean result with that above given was very striking, and in one analysis, using three and one-half grammes of carefully selected material, we obtained 66.662% Bromine.

to produce all the variation from the normal composition which the above results present. A simple calculation will show that an absorption of only $\frac{47}{1000}$ths of one per cent of oxygen, or less than half a milligramme by each gramme of iodide of antimony, would reduce the per cent of iodine from the theoretical value, 76.047, to the mean of the above results, 75.832; and that a corresponding absorption of three-quarters of a milligramme would reduce the per cent to 75.700, the lowest observed. It is not, therefore, surprising that we could obtain concordant results only with material which had been both purified by crystallization and also *recently* sublimed.

Returning now to discuss again the cause of the disagreement of the analyses of antimonious chloride with our otherwise consistent results in regard to the atomic weight of antimony, it was obvious that the strong hygroscopic power of the chloride must lead to a replacement precisely similar to that which is produced in the iodide by direct oxidation; for, as we have before said, the crystals of antimonious chloride cannot be exposed to the atmosphere for an instant without absorbing a perceptible amount of moisture, and every molecule of water thus absorbed reacts on a molecule of the chloride, thus: —

$$SbCl_3 + H_2O = SbOCl + 2\,HCl.$$

And when the antimonious chloride is boiled, the hydrochloric acid formed is given off, while the oxichloride remains behind, dissolved in the great mass of the liquid. Indeed, it seems impossible, with our ordinary appliances, to prepare or purify antimonious chloride without its becoming contaminated with oxichloride; and our experiments would indicate that when once it has been formed, as above described, in the mass of the material, it cannot be wholly removed by distillation or crystallization, however often these processes may be repeated.

Naturally, our attention was very early called to this obvious source of impurity in the antimonious chloride we prepared; and we noticed from the first that, even after the material had been many times distilled, there was always left, on repeating the process, a very small amount of dark-colored residue. We had examined the residue, and found that it was a mixture of chloride and oxichloride of antimony, colored by a trace of carbonaceous material; and we had made a long series of analyses for the purpose of studying the effect produced by the action we have described. The result of these analyses is given in the following table. We started with material already purified by fractional distillation and crystallization, and distilled it ten times in succession; not, however, carrying the distillation to absolute dryness, but leaving, so far as we could judge by the eye, about the same amount of residue in the retort each time. These residues we analyzed, as we did also the final distillate. The material first distilled was the same as that marked *c* in the table on page 40, and we assumed that the average of the results there given truly represented its composition.

ANALYSES OF ANTIMONIOUS CHLORIDE.

RESIDUES AND DISTILLATES.

	% of Chlorine.
The original purified preparation	46.64
The residue of 1st distillate	45.71
" 2d "	45.66
" 3d "	46.03
" 4th "	46.26
" 5th "	46.26
" 6th "	46.00
" 7th "	46.03
" 8th "	45.94
" 9th "	45.65
" 10th "	45.99
The last distillate	46.62

Although, under the circumstances, we could not expect great precision, yet it was evident from these analyses that the amount of impurity in the residues was not diminished by the successive distillations; and we therefore concluded that additional oxichloride of antimony must be formed each time during the very short contact with the atmosphere which the transfers between the several distillations necessarily involved. But, on the other hand, the very remarkable fact that these ten distillations produced no sensible change in the composition of the great mass of the material seemed to indicate equally clearly that this action of the atmosphere had no perceptible influence on the final result; and this opinion was still further strengthened when, on twice distilling portions of the last distillate, at a low temperature, in a current of dry hydrogen, we obtained products giving again — very nearly at least — the same per cent of chlorine. And, lastly, when to all this evidence were added the results of the complete analysis of the chloride, showing an amount of antimony which fully supplemented the very constant per cent of chlorine, the assumption that any material amount of impurity could be present appeared wholly untenable. Yet we have seen how this assumption was forced back upon us by the subsequent results of the investigation.

Returning to the subject after our experiments with iodide of antimony, we, for the first time, fully appreciated how very small an amount of oxygen — the only real impurity present — was required to reduce the per cent of chlorine in antimonious chloride from 47.02, the amount corresponding to Sb = 120, to 46.61, which corresponds to Sb = 122; for, while the effect is so differently produced, yet the result of the action of the atmosphere on the chloride of antimony is wholly like that of its action on the iodide. It ends in replacing a small amount of chlorine by oxygen; and although, in consequence of the smaller atomic weight of chlorine, it requires in this last case a larger replacement to produce a corresponding change of percentage composition, yet still the amount required to make all the difference in question is very small; so that, when we come to sum up the supposed completed results (as on page 42), it might easily be covered up by slight inaccuracies of the analytical work. An easy calculation will show that the substitution of but $\frac{146}{1000}$ of one per cent of oxygen for the equivalent amount of chlorine would reduce the per cent of this last element in the chloride from 47.020, corresponding to Sb = 120, to 46.608, which corresponds to Sb = 122; and such a substitution would result from the absorption of only $1\frac{6}{10}$ milligrammes of water by each gramme of the chloride. The composition of the material would then be as follows: —

COMPOSITION OF ANTIMONIOUS CHLORIDE WITH $\frac{146}{1000}$ % OF O WHEN CL = 35.5 AND SB = 120.

Chlorine	46.608
Oxygen	.146
Antimony	53.246
	100.000

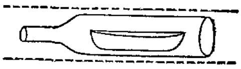
Now it will be seen by referring to the tables, on pages 40 and 42, that these percentages do not differ from the mean of the results of our previous analyses as much as these results differ among themselves; and we therefore determined to repeat these analyses, hoping that the experience we had acquired in both chlorine and antimony determinations would now enable us to obtain results sufficiently sharp to show even the small differences of composition which the substitution in question would produce.

Meanwhile, we instituted a series of experiments with a view of studying the decomposition which the oxichloride of antimony undergoes under the action of heat, in the hope that we might thus discover some method by which the amount of oxichloride of antimony in our preparations might be directly determined. For this purpose, we used first crystallized SbOCl, obtained by the action of alcohol on chloride of antimony in a sealed tube, which we weighed out into a platinum nacelle, and heated to various regulated temperatures, using for this purpose the apparatus already described. It appeared that the decomposition took place in two stages. The first stage of the decomposition began between 167° and 175°, but was not completed until between 260° and 280°. The second stage began at about 320°, but required for its completion a red heat. During both stages, chloride of antimony sublimed; and there was left in the nacelle at the close of the process beautiful crystals of Sb_2O_3. In another experiment, we used crystallized $Sb_4O_5Cl_2$, prepared in the same way as the SbOCl, but with different proportions of alcohol and chloride of antimony. In this case, the decomposition did not begin until 320°, but in other respects both the process and the products were as in the first experiment. It was quite evident that the chemical changes which took place in the two stages of decomposition we have noticed were represented by the following reactions:—

$$\text{First stage:} \qquad 5\ SbOCl = Sb_4O_5Cl_2 + SbCl_3\,; \qquad (1)$$

$$\text{Second stage:} \qquad 3\ Sb_4O_5Cl_2 = 5\ Sb_2O_3 + 2\ SbCl_3\,; \qquad (2)$$

but the relative weights observed in the first two experiments were of no value, because it was evident that a no inconsiderable amount of Sb_2O_3 was lost by sublimation. Since, however, the small sublimate of oxide condensed in the glass combustion-tube very much nearer the nacelle than the very much larger sublimate of chloride, we varied the apparatus in our third experiment so far as to place the nacelle in a tube of the shape represented in the accompanying figure. This tube was weighed with the nacelle, and was so selected that it quite closely fitted the combustion-tube within which it was placed for heating, as shown in figure by dotted lines. And it is evident that, while with this arrangement the $SbCl_3$ would be swept



by the CO_2 gas into the colder portion of the combustion-tube, the greater part at least of the sublimed oxide would be retained in the small tube, which was of course at each stage weighed with the nacelle as at first. Our results were as follows: —

Weight of SbOCl	0.4939	grammes.
Loss at 280°	0.1271	"
Required by theory of reaction 1, if Sb = 120	0.1305	"
Total loss at red heat; that is, in both stages .	0.2179	"
Required by theory of reactions 1 and 2 . .	0.2174	"

It was evident from this determination that the order of the decomposition was precisely that indicated by our reactions, although the end of the first stage was not quite so sharply marked as the end of the second; and this would naturally be expected.

As the residues obtained on distilling chloride of antimony showed, when further heated, precisely the same order of phenomena which we have just described, and when heated to redness yielded the same crystals of oxide of antimony as before, it was plain that the residue left on evaporating the chloride at a temperature not exceeding 120° was chiefly at least SbOCl; but that this when heated more intensely was converted into $Sb_4O_5Cl_2$ before the temperature reached 280°, and finally at a red heat was converted wholly into Sb_2O_3. We therefore endeavored to determine the amount of oxichloride in one of our preparations of chloride of antimony by distilling a weighed amount from a platinum nacelle at as low a temperature as possible in a current of dry carbonic acid, and heating the residue to a temperature of about 275°. We thus obtained the following results: —

No.	Wt. of $SbCl_3$.	Residue.	% of Residue $Sb_4O_5Cl_2$.
1.	6.7286	0.0212	0.315
2.	4.5150	0.0151	0.334
3.	7.9320	0.0258	0.325

In order to yield 0.146 % of oxygen, which would reduce the per cent of chlorine in the preparation from 47.020 to 46.608, as in the scheme on page 62, there would be required 1.155 % of $Sb_4O_5Cl_2$.

Although the results of the above determinations accord within a few per cent of the quantity estimated, yet it was perfectly clear during the course of the experiments that they did not at all represent the total quantity of the oxichloride present in the preparation examined. Not only was the composition of the preparation not materially altered by the slow distillation, — a fact shown by the determinations marked *e* in the table on page 40, and by which we were misled at the outset, — but also the product from our distillation yielded when distilled again apparently as much residue as before. In a word, we found the same phenomena repeated in these distillations at a low temperature which had been so noticeable when the chloride was distilled at its boiling point, and which are so strikingly illustrated by the results given on page 61. It is possible, as before suggested, that the effects might arise from a small additional absorption of water at the successive transfers which the repeated distillations involved; or, in the later experiments, from the circumstance that the very extensive apparatus employed for drying the carbonic dioxide was not completely effectual. Still, now that our attention had been called to the danger, and we had taken unusual precautions on both these points, the explanations suggested did not seem to us sufficient; and we came to the conclusion that the oxichloride must distil over with the chloride of antimony to a certain limited extent, and that it was only an excess above this definite amount which was left behind as residue. Of course, SbOCl not only is not volatile, but is at once decomposed by heat; and we do not suppose that this compound by the tension of its own vapor is carried over in distillation. It is a very dilute solution, as it were, of SbOCl in $SbCl_3$ which thus distils; and the distillation of the oxichloride may resemble the carrying over of boracic acid by the vapor of water, and similar phenomena, the result, as it is has always appeared to us, of a feeble kind of chemical union which has been usually designated by the term "molecular combination." Such a theory would account for the remarkable constancy which we have found in the chlorine determinations of the various preparations of antimonious chloride purified by distillation. But, on account of the very great difficulty of removing all possible disturbing causes, we found it impossible to obtain a rigid experimental demonstration of our theory without much more time and labor than we could then command. We hope to return to the subject hereafter. Meanwhile, however, it was evident that we could place no reliance whatever on the results just obtained. Nevertheless, the determinations were of value on account of the contrast between these results and those of a similar series of experiments on the residues from antimonious bromide which we collect in the following table: —

No.	Wt. of $SbBr_3$.	Residue chiefly $Sb_4O_5Br_2$.	% of residue.
1.	2.8342	0.0010	0.035
2.	2.0220	0.0006	0.030
3.	4.6730	0.0010	0.021

As will be seen, this residue is less than one-tenth of that obtained from the chloride, and is practically insignificant. Evidently, then, in the determination of the atomic weight of antimony more accurate results may be expected from the analysis of the bromide than from the analysis of either the chloride or the iodide of this element. The intermediate position of the bromide renders it, in a very remarkable way, the most stable of the three compounds. It absorbs moisture far less eagerly than the chloride, and it absorbs oxygen far less readily than the iodide, and is thus in great measure protected against each of these two sources of the same impurity.

We come finally to the new analyses of antimonious chloride we had undertaken. Fortunately, some of the old preparation that had been distilled so often had been preserved. It had been boiled for a long time since the last analyses were made, and kept in the same flask used for determining its boiling point, which had stood meanwhile tightly corked in a desiccator over sulphuric acid. The solid mass in the flask was easily broken up without exposure to the air by simply heating it to the melting point, and shaking it in the flask as soon as, beginning to melt, the mass had separated from the glass. Near its melting point, chloride of antimony becomes very friable, and is thus easily reduced to coarse powder, whence probably the old alchemistic name of butter of antimony. It is also worthy of notice that neither the bromide nor the iodide acts in this way, as we found out in more than one instance to our cost.

Thus we were readily able to prepare our material for analysis, and, by a thorough mixing of the mass, to insure that the several samples taken had a uniform composition. In regard to the antimony determination, no further details are necessary. It was conducted, as described before, with every minute precaution which experience had suggested; and we give the full details, in order to show how completely we had been able to overcome the difficulties which it at first presented, and we feel confident that there is no process of wet analysis which is capable of giving more accurate results than this.

DETAILS OF ANTIMONY DETERMINATION.

The antimonious chloride was first transferred to a very carefully dried weighing tube, and thence to the large flask in which it was dissolved. The transfer to the weighing tube was made in a dry atmosphere, and only required two or three seconds. It is evident, however, that a slight absorption of moisture at this point is not important; for, even if it increased the apparent weight of the assay by several milligrammes, it would only reduce to a barely perceptible extent the percentages of all the constituents leaving the relative values wholly unchanged. It is only when, on boiling the chloride, after such an absorption, the chlorine is driven off, that the essential change of composition results.

Weight of tube and antimonious chloride	20.9609	grammes.
„ „ after transfer to flask	16.3920	„
„ chloride analyzed	4.5689	„

The weight of the tube and chloride while on the balance pan remained invariable for a sufficient length of time to give positive assurance of the constancy of the weights. The chloride was dissolved in a saturated solution of tartaric acid containing about 15 grammes of the pure acid, and then diluted with carbonic acid water and precipitated as before described. The precipitate, having been washed and collected as before, was dried in an air bath, at about 110°.

Weight of small filter	0.0434	grammes.
„ porcelain crucible	101.2132	„
	101.2566	„
„ crucible and precipitate	104.6762	„
„ red sulphide of antimony	3.4196	„

A portion of the dried precipitate dissolved in hydrochloric acid gave no residue. The rest was then transferred to a platinum nacelle, and heated, as has been described, in a current of dry carbonic dioxide gas. No sublimate was formed, and only a very slight empyreumatic odor could be perceived.

Weight of platinum nacelle	6.2493	grammes.
„ nacelle and dried precipitate	9.5273	„
„ portion taken	3.2780	„
„ nacelle and precipitate after heating to 285° for over half an hour	9.5234	„
Loss of weight of portion taken	0.0039	grammes.
Corresponding loss for whole precipitate	0.0041	„
Weight of red sulphide as above	3.4196	„
„ gray sulphide	3.4155	

The carbonaceous residue left on dissolving this whole amount of gray sulphide in hydrochloric acid was barely perceptible. It was collected, however, as usual, on a weighed paper disk, and estimated.

Weight of small paper filter	0.0198	grammes.
„ same with residue	0.0212	„
„ residue	0.0014	„
Calculated for whole precipitate	0.0015	„
Weight of gray sulphide as above	3.4155	„
Total weight of gray sulphide	3.4140	„
Corresponding weight of antimony assumed to be $\frac{5}{7}$ of the sulphide	2.4386	„
Per cent of antimony in the antimonious chloride under examination	53.374	„

It will be noticed that this result is practically identical with the mean of the previous determinations, which, as will be seen by reference to the table on page 42, was 53.401; and, by reviewing the facts stated in that connection, it will be perceived that this agreement is in itself alone a strong confirmation of the conclusion which we deduced from our first experiments on the synthesis of the gray sulphide of antimony, — that of the two values of the atomic weight of antimony in question, the lower is the more exact.

Coming next to the chlorine determinations, we noticed, for the first time, an effect which, under certain circumstances, may have an important influence on the accuracy of this well-known process, as employed in the analysis of chloride of antimony. In a precipitate of argentic chloride that had been deposited from an unusually concentrated solution of antimonious chloride in tartaric acid, and had stood over night, our attention was called to some crystalline grains, which, on examination, proved to be a compound of tartaric acid, antimony, and silver. We soon found that this product could be readily obtained by concentrating the filtrate from the precipitate of argentic chloride, and adding to it, while still warm, an excess of argentic nitrate. On cooling, the new crystals form in abundance. They have not yet been measured, but under the microscope they have the general aspect of right rhombic plates or prisms, with hemihedral modifications, — a general form which is so characteristic of the tartrates, and which we ourselves have previously studied in our crystallographic determinations of the tartrates of rabidium and cæsium.* We obtained for the amount of silver in the crystals, as a mean of three analyses, 26.30 per cent. The compound $Ag,SbO,H_2 \equiv O_4 \equiv (C_4H_2O_2) \,.\, H_2O$ would require 26.34 %. The crystals may therefore be regarded as tartar emetic, in which the potassium has been replaced by silver; and they resemble the crystals of this well-known salt in general form. They are evidently the same substance obtained by Wallquist† by precipitating nitrate of silver with tartar emetic, and analyzed both by him and by Dumas and Piria. These chemists obtained respectively 27.31 and 28.05 per cent of oxide of silver, which corresponds with the result given above as closely as could be expected; but they appear to have prepared the substance only in an amorphous condition. At least, in the description quoted, no mention is made of any crystalline form.

These crystals of argento-antimonious tartrate are apparently not acted upon in the least by cold water, and only slightly by boiling water; and finding this very insoluble material mixed with the precipitated chloride of silver, under the conditions stated, we were led to fear that it might be occluded to some extent by this precipitate, even when formed in much more dilute solutions of antimony and tartaric acid. The phenomenon was very similar to that we had already studied in the occlusion of the oxichloride by the sulphide of antimony; and there was reason to fear that, as in the previous case, an occlusion of this double tartrate might result, even when the substance would not otherwise be precipitated. How far such an action could have vitiated our previous results, it was, of course, now impossible to determine; but, as we previously stated, we had always taken great care not to add more than the slightest possible excess of argentic nitrate, and this was especially true in our more recent determinations. Now, however, we were on our guard, and in the following determinations very great pains were taken to add just the requisite amount of the silver salt, and the argentic chloride was subsequently examined for traces of any such occlusion. But, excepting this close attention to well-known precautions, the determinations were made in the same way as before.

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* Am. Jour. of Science and Arts. (2), xxxvii. 70.

† Gmelin Handbook, Cavendish Edition, x. 326.

ANALYSIS OF ANTIMONIOUS CHLORIDE.

No.	Wt. of $SbCl_3$.	Wt. of AgCl.	% of Chlorine.
1.	2.2220	4.1682	46.407
2.	1.9458	3.6512	46.420
Mean value			46.413

Bringing now the results together, — estimating the amount of oxygen by difference, as is usual in chemical analysis, and calculating what would be the composition of a preparation of antimonious chloride in which $\frac{213}{1000}$ of a per cent of oxygen had replaced an equivalent amount of chlorine, assuming, of course, Sb = 120 and Cl = 35.5, — we obtain the following very striking accordance: —

	Analysis.	Theory. Sb = 120, Cl = 35.5.
Chlorine	46.413	46.418
Oxygen	.213	.213
Antimony	53.374	53.369
	100.000	100.000

The general conclusions, then, which we deduce as the results of this investigation, are —

First, that the value of the atomic weight of antimony found by Schneider in 1856 — Sb = 120.3 — must be accurate within a few tenths of a unit, but that the most probable value of this constant, as deduced from our experiments, is Sb = 120, when S = 32.

Secondly, that the apparent disagreement with this result, presented by the partial analyses of antimonious chloride, is *probably* due to the constant presence of oxichloride in the preparations of this compound.

The investigation from the first has been a study of constant errors; and those who have followed us through the details will certainly allow that the opinions expressed at the beginning of this paper (on page 9) were not hastily conceived, even if they do not fully agree with our conclusions. In the attempts to correct or balance such errors, we have found at once the chief difficulties and interest of our work, and the secondary results thus reached seem to us the most important fruit of the whole investigation. Seeing, then, the sources of constant error we have discovered, and knowing that there are others whose influence we have been able to trace, although we have not been able to define them as clearly as we could desire, it would be presumptuous in us to express too great confidence either in the correctness of our theories or even in the conclusiveness of our experimental results. Of this, however, we feel assured, that more trustworthy results cannot be expected from a repetition of the same processes until a more complete and accurate knowledge has been acquired of the substances employed. We have therefore proposed to ourselves a more thorough investigation of the haloid compounds of antimony, and the first results of this investigation we shall shortly publish. After the requisite data have been thus collected, we hope to return to the old problem with such definite knowledge of the relations involved as will enable us to obtain at once more sharp and decisive results than are now possible.

During the course of this investigation, we have been successively aided in the experimental work by Dr. F. A. Gooch, Mr. C. Richardson, and Mr. W. H. Melville, at the time students in this laboratory; and without their assistance we could not have accomplished the great amount of labor it involved.

HARVARD COLLEGE LABORATORY, *June* 12*th*, 1877.

II.

RE-EXAMINATION OF SOME OF THE HALOID COMPOUNDS OF ANTIMONY.

By Josiah P. Cooke, Jr.,

Erving Professor of Chemistry and Mineralogy in Harvard College.

Our chief object in this paper is to describe some remarkable crystallographic and chemical relations of antimonious iodide, first noticed during the investigation of which an account has just been given; but we will also take the opportunity to give the results of some observations upon antimonious chloride and antimonious bromide, as well as upon the oxichlorides, oxibromides, and oxi-iodides of antimony, all of which have more or less bearing on the principal subject.

Antimonious Chloride ($SbCl_3$).

Very perfect and brilliant crystals of antimonious chloride can be made in one of two ways, and both methods yield crystals of the same general form and habit.

The first method consists in cooling a saturated solution of the chloride in carbonic disulphide. Antimonious chloride is very soluble in this liquid, when near its boiling point; but the solubility diminishes very rapidly with a falling temperature, and, when the solution is cooled with a freezing mixture, by far the larger part of the substance crystallizes out. During our experiments, we frequently noticed, with these solutions of antimonious chloride, the phenomena of supersaturation. A solution saturated at the boiling point of the solvent, and cooled in a clean glass flask, may, if undisturbed, remain liquid for an indefinite time; but, the moment a bit of the solid substance is dropped in, the crystals form with great rapidity, and the very marked rise of temperature which we have observed under these circumstances indicates that the crystallization is attended with the liberation of an unusually large amount of latent heat. Similar phenomena of supersaturation we noticed with solutions of antimonious bromide, which, although less soluble than the chloride, dissolves very freely in the same solvent;

but, with the solutions of the far less soluble antimonious iodide (in disulphide of carbon) the phenomena were not perceptible. It is evident, from these facts, that the phenomena of supersaturation are not confined to aqueous solutions, or to substances which take into their crystalline structure a portion of the solvent, like water of crystallization.

The second method of obtaining crystals of antimonious chloride, is the familiar process of pouring off the still fluid portion, after the melted substance has partially solidified. Since, in consequence of the low temperature at which it hardens and the large amount of latent heat evolved in the process, the chloride *sets* comparatively slowly, the crystals form under these circumstances with great perfection, and are left clear and brilliant when the fluid is poured off.

It is very easy to obtain, by either of these methods, very perfect crystals, with very brilliant faces; but to measure these crystals is a difficult problem, which we have as yet been able to solve only imperfectly. Antimonious chloride is so very hygroscopic, that, during the short time required to isolate the crystals and mount them in tightly corked glass-tubes (and under such protection the measurements were made), the faces so far lost their lustre as to render the reflected image of the goniometer signal indefinite. Hence, the results given below are to be regarded as only approximate, and may be in error to the extent of even a degree. The angles of the vertical prism are probably the most accurate, because the crystals could be most quickly mounted with this dome parallel to the axis of the tube. The two domes present were measured on different crystals, and the angles given are the results of what were considered the most favorable observations. When the crystal was once mounted, its position could not be shifted; for the antimonious chloride attacked the wax used, and this circumstance added to the difficulties attending the necessary manipulations in measurements with the goniometer.

The crystals of antimonious chloride are trimetric, and have the same general habit whether obtained by the one or the other of the two methods just described. The chief difference is a greater or less degree of elongation in the direction selected as the vertical axis, — a difference which is shown by Figs. 1 and 2 of Plate I. The crystals were examined with a polarizing microscope arranged as a stauroscope; and we observed that, when the crystals were resting on either of their planes, so that the light passed between two opposite and parallel surfaces, the principal optical sections were, as nearly as could be observed, either parallel or normal to the prismatic edges. This, although not conclusive evidence in itself, confirms the conclusion in regard to the crystalline system, which was based on the symmetry of the external form.

CRYSTALLINE FORM OF ANTIMONIOUS CHLORIDE.

ORTHORHOMBIC SYSTEM.

Forms {110} and {011}

Figs. 1 and 2, Plate I.

$a = 1.263 \quad b = 1 \quad c = 1.109$

Angles between normals.

110 on $1\bar{1}0 = 103° 16'$

011 „ $0\bar{1}1 = 115° 57'$

In order to obtain the specific gravity of antimonious chloride in the solid state, we filled a specific-gravity bottle nearly full of the melted substance; and, after the mass had "set," we added (so as to completely fill the bottle) some rock oil, which had been rectified over sodium. We then kept the bottle in the exhausted receiver of an air-pump, long enough to remove any entangled air; and, finally, after inserting the ground stopper and wiping away the excess of oil, we took the weight at a carefully regulated temperature. We could find no liquid on which antimonious chloride does not act, to a greater or less extent. It acted slightly even on this rectified rock oil, although only very slowly; so that, by working as quickly as possible, we must have obtained a result which was at least nearly accurate. We used the same preparation of antimonious chloride, of which a complete analysis is given on page 70 of the previous paper. The weight taken was 19.9575 grammes, which displaced 5.0212 grammes of oil. The specific gravity of the rock oil at 26°, referred to water at the same temperature, was 0.7693; and we found —

Specific gravity of Antimonious Chloride at 26°, referred to rock oil at same temperature . .	3.976
Specific gravity of Antimonious Chloride at 26°, referred to water at same temperature . . .	3.064

The melting point of antimonious chloride was determined by observing the constant temperature during the slow crystallizing of a considerable mass of the melted substance, the liquid being stirred meanwhile with the bulb of the thermometer, which was immersed up to the division on the stem marking 15°. The preparation used in this determination was that designated by *f*, in the table on page 40; and we obtained, as the

Melting point of Antimonious Chloride, 72° C.

We obtained also, and in the usual way, for the

Boiling point of Antimonious Chloride, 216° C.

In several instances, while rectifying this substance as described in the last paper, we followed the boiling point, and observed that it was constant, during the whole period of the distillation.

ANTIMONIOUS BROMIDE ($SbBr_3$).

The methods used for preparing and purifying the bromide, as well as the chloride, of antimony, have been so fully described in the previous paper that no further details are necessary here. We obtained very brilliant crystals of the bromide, not only by the two methods described under the last head, but also by sublimation with the apparatus represented on page 57 of this volume. As treated in either of these ways, the habit of the substance is to form needle-shaped crystals, which run out into fine points without definite terminations, and often group themselves into irregular bundles, — a very common feature of this type of crystals. Only on one occasion (then by slowly cooling a solution in disulphide of carbon) did we obtain well terminated crystals; and, although we afterwards tried again and again, we have not yet been able to reproduce them. Unfortunately, moreover, before we were ready to make our measurements, the small terminal planes of these crystals had already become tarnished by the atmosphere. For, although the substance is so much less hygroscopic than antimonious chloride, yet the crystals of antimonious bromide soon lose their sharpness, if exposed even to what we call our dry winter air. Hence, we were not able to measure the angles between the terminal planes with a reflective goniometer. The approximate value of 101 on 100, we obtained by measuring the corresponding edge angle under the microscope, and by frequent repetitions of the measurement on different crystals, or on the two sides of the same crystal, securing as great accuracy as is possible under such circumstances; but the result cannot be relied upon within one or two degrees.

Crystalline Form of Antimonious Bromide.

Orthorhombic System.

Forms {100}, {010}, {101}, {110}, {111 (?)}

Fig. 3, Plate I.

$a = 1.224 \quad b = 1 \quad c = 1.064.$

Angles between normals.

010 on 110	39° 14′

Also, measured with microscope.

101 on 100	49° approximately

Calculated.

101 „ $\bar{1}01$	82°
111 „ $11\bar{1}$	72° 4′
111 „ 010	38° 47′
111 „ $\bar{1}11$	61° 30′

The octahedral angles were calculated on the assumption that the observed planes were those of a fundamental octahedron; but, although the intersections with 110 appeared to be parallel, yet the edges were too indefinite to give any certainty on this point.

We also examined the crystals of antimonious bromide with the polarizing microscope, and observed that one of the principal optical sections was parallel to the prismatic edge, whether the light passed normal to one or the other of the two pinacoids 010 and 100. *

The specific gravity, in the solid state, of purified antimonious bromide was taken in precisely the same way as that of antimonious chloride: —

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* The only previous description of these crystals of which we have any knowledge was given by Nicklès, "Comptes Rendus," XLVIII. 837, in these words: "Le bromure d'antimoine se présente en otaèdres rhomboidaux, parfois modifiés par des faces terminales; ils constituent alors des prismes aplatis de 69 degrés terminés par des pointements de 80 degrés; l'angle de deux faces contigués de l'octaèdre est de 181 degrés (les minutes ont dû être negligées le cristal étant trop déliquescent)." In a later paper, "Journal de Pharmacie et de Chimie" (3), XLI. 142, a figure is given, and this description is essentially repeated, correcting the obvious misprint, 181 for 131; but, nevertheless, this error has been very generally copied. The crystals measured by Nickles must have had a very different habit from any we have seen, and we have not been able to reconcile his description with our own observations.

Weight of Antimonious Bromide taken . .	32.2938 grammes.
Specific gravity referred to Kerosene at 23° .	5.386
„ „ Water at 23° .	4.148

By the same methods used with antimonious chloride, we made several determinations of both the melting and the boiling points of purified antimonious bromide, with the following results:—

Melting point of Antimonious Bromide .	93° C.
Boiling point „ „ .	280° C.

Antimonious Iodide (SbI_3).

There are three crystalline conditions of antimonious iodide,—the hexagonal, the orthorhombic, and the clinorhombic or monoclinic, of which only the first has hitherto been described.

Hexagonal Antimonious Iodide.

Hexagonal crystals of iodide of antimony of a deep ruby red color can be readily obtained, either by cooling or by evaporating a saturated solution of this substance in disulphide of carbon, and this solution is easily prepared (as described in the previous paper) by shaking up, with finely pulverized metallic antimony until the color is discharged, a strong solution of iodine in the same solvent. These crystals were described by Nicklès, in connection with those of antimonious bromide, in the paper just referred to. He states that they are hexagonal, double pyramids, with a basal angle of 133°; and his description is referred to by Schneider,* who also obtained hexagonal crystals, both from the disulphide of carbon solution, and also by subliming a mixture of antimonious sulphide with iodine. Schneider speaks of the crystals as brilliant, sharp, six-sided leaves or tables; but gives no additional measurements. All the crystals which we have examined (and we have seen the products of a great number of crystallizations) are combinations of a rhombohedron with its first obtuse rhombohedron and the basal planes, as shown in Figs. 4 and 5, Plate I. The habit of the crystals, however, differs very greatly with the conditions under which they are formed. When deposited during the rapid cooling of a solution saturated at the boiling point of the very volatile solvent, they are, as Schneider states, small and leaf-like, with a very definite hexagonal outline; but still, when seen with a microscope by reflected light, the rhombohedral planes can be distinguished on the edges. When formed during the slow evaporation of the solvent, they are, as Schneider also noticed, larger and more tabular. We ourselves have further observed, that, when the solution contains an excess of iodine, the rhombohedral planes become much more dominant, and the crystals greatly elongated in the direction of the vertical axis. The basal plane is then often reduced to a small triangular face; and we have seen crystals in which it had almost, if not wholly, disappeared. Among such crystals, we have observed macles hemitroped on the basal section; and the rhombohedral planes are frequently strongly striated parallel to the basal edges.

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* Poggendorff, Annalen, cix. 610, 1860.

Under circumstances similar to those just mentioned, especially when the amount of free iodine in the solution is proportionally large, the crystals frequently group together into stars with six rays. These rays are formed by crystals elongated in the direction of one of the diagonals of the hexagonal section, and each by itself has the outward aspect of the trimetric system. The rays often branch, but in all cases at the constant angles of 60° or 120°; and the whole group preserves a more or less regular hexagonal outline. Such groups may be regarded as skeleton crystals, and their formation is probably determined by a deficiency of the substance of the crystals in the mixed solution from which they are formed. The polariscope shows that they have throughout an hexagonal structure, and their formation indicates a tendency in the crystals of this substance (often manifested in single crystals to a less degree) to excessive development in a single direction, thus imitating a trimetric habit. As we shall hereafter see, this habit is not without its significance.

When iodide of antimony is sublimed as described in the previous paper (page 57), and also by Schneider (*loc. cit.*), it condenses in very broad thin leaves or plates, which hang from the surfaces of attachment by their edges. Even these, however, frequently exhibit on their free edges, not only the hexagonal outline, but also the rhombohedral planes; and the polariscope shows that the surfaces of the leaves are simply widely extended basal planes.

Iodide of antimony is not hygroscopic, and for this reason the crystals present conditions which are more favorable for accurate measurements than the crystals either of the chloride or of the bromide of the same element. Nevertheless, our results were not as constant as the brilliancy of the crystals led us to expect; and we met with variations in the angles which we could not ascribe solely to imperfections of the faces or to other causes of inexactness in the measurements. The uncertainty thus arising does not, however, amount to more than a few minutes. We give the results obtained with the most perfect crystals we could find, on which we were able to measure all the angles of the principal dome between the two basal planes.

CRYSTALLINE FORM OF ANTIMONIOUS IODIDE.

HEXAGONAL VARIETY.

Forms {111}, {100}, {011}

Figs. 4 and 5, Plate I.

Angles measured between normals.

111 on 100	72° 28′
100 „ $0\bar{1}\bar{1}$	49° 22′
$0\bar{1}\bar{1}$ „ $\bar{1}\bar{1}\bar{1}$	58° 8′
	179° 58′

These measurements correspond to the dimensions of a modified rhombohedron in which the axes of Miller's system make with each other the angle of 54° 40′, or the vertical axis of Naumann's system has the value $C = 2.769$. The crystals cleave readily parallel to the basal plane.

Angles between normals.	Calculated.	Measured.
111 on 100	72° 38′	72° 28′
111 „ 110	49° 23′	49° 22′
110 „ $00\bar{1}$	57° 59′	58° 8′
100 „ 010	111° 30′	
110 „ 011	94° 28′	
100 „ 110	55° 45′	

The crystals are optically uniaxial with very strong negative double refraction, and the broad plates obtained by sublimation furnish excellent objects for the polariscope; but such preparations are not durable if exposed to the atmosphere, for a reason which will appear further on. The rings of the interference figure, as seen by common light, are nearly black, but with strong-colored fringes, — red on the inside, and greenish yellow on the outside. This is a natural result of the selective absorption of this highly colored medium; but the effect is, nevertheless, very striking.

Evidently, then, so far as yet appears, the iodide of antimony is not isomorphous with the corresponding bromide and chloride, although there is no group of compounds which we should by analogy expect to find more closely isomorphous than the chloride, bromide, and iodide of the same element. It has been shown, however, by other crystallographers as well as by ourselves, that the hexagonal forms are closely related to the trimetric, and that when the angle of the rhombic prism becomes equal to or even closely approaches 60° or 120°, the last may imitate, if they do not actually assume, both the external aspect and internal structure of the first. We were, therefore, led to suspect that we had before us another example of such a relation, and that the iodide of antimony might thus be constructively isomorphous with its allied compounds. The following calculation, moreover, strongly sustained this view: —

The large development of the pinacoid planes (010 and $0\bar{1}0$) which is very characteristic of the crystals of bromide of antimony, besides the general form of these crystals (as shown by our figure), indicate very clearly that these pinacoids are the analogues of the basal planes of the hexagonal crystals of iodide of antimony; and, if so, then the isomorphism of the bromide and the chloride indicates that, for the last compound, the analogues of these planes would be a corresponding pair of pinacoids which does not appear on the actual crystals. Again, we find the analogue of the angle, 120°, on the hexagonal section of the crystals of antimonious iodide, in the angle 98°, between the planes 101 and $\bar{1}01$ on the crystals of antimonious bromide; and although the corresponding planes do not appear on the crystals of antimonious chloride, yet the equivalent angle can be easily calculated, and will be found to be 97° 26′. If now we compare the tangents of the halves of these last two angles with $\frac{2}{3}$ of the tangent of 60°, we shall obtain the following relations: —

For $SbCl_3$ tang. 48° 43′ = 1.139
„ $SbBr_3$ tang. 49° = 1.150
„ SbI_3 $\frac{2}{3}$ tang. 60° = 1.155

If, further, we take into consideration the third axis, this relation will appear still more close and simple. Making, then, in the crystals of antimonious chloride and bromide the half-axis *c* our unit, and regarding for the time, as the vertical axis, the half-axis *b*, which corresponds to the vertical axis of the hexagonal form, giving also to this last axis its known value, — we obtain the following comparison between the axial dimensions of the two orthorhombic forms and of the hexagonal form, if referred to corresponding orthorhombic system of axes: —

$$\begin{array}{llll}\text{For } SbCl_3 & a = 1.139 & \acute{b} = 0.902 & c = 1 \\ \text{,, } SbBr_3 & a = 1.150 & \acute{b} = 0.940 & c = 1 \\ \text{,, } SbI_3 & a = 1.155 & \frac{1}{3}\acute{b} = 0.923 & c = 1\end{array}$$

It thus appears that, constructively, iodide of antimony is closely isomorphous with the two allied compounds, and the small apparent differences in the axial dimensions of the three forms are no greater than the uncertainties in these values themselves arising from the imperfect measurements of some of the angles.

But although the rhombic prism of 60° or 120° may imitate the external aspect of an hexagonal form, as is frequently the case with the micas and vermiculites, yet (as the optical relations of these very minerals show) the two classes of forms may still remain perfectly distinct; and it does not, therefore, by any means follow that the rhombic prism passes into the hexagonal system when the prismatic angle becomes 120°. We have, however, shown, in the paper just referred to, that, by a species of interlamination, — interlaminar macling, we may name it, — the orthorhombic crystals of foliated minerals frequently imitate the structure, as well as the forms, of the hexagonal system; and although there is an obvious distinction between such structures and homogeneous crystalline masses, like calcite or the substance we are considering, yet analogy would suggest, that, even the true hexagonal structure may result from a more fundamental macling of the same kind; and we advanced the theory, that it might be the result of what we called molecular macling. According to this theory, the *crystalline* molecules of hexagonal forms are, in some cases if not in all, groups of three simpler molecules, each of which (so far as its chemical constitution is concerned) is a unit in itself, and possibly under certain conditions may act as a unit in a crystalline structure, and probably always becomes isolated when the substance is converted into vapor. Our theory also assumes that the members of these groups are united among themselves in the same relative positions as the diagonals of a regular hexagon, so that the optically uniaxial character of hexagonal crystals is an effect of such grouping, and the hexagonal form an obvious result of the juxtaposition of the six-sided groups. Further, we suppose that the simpler molecules are of such a nature that, when united as individuals in positions parallel to each other, they would form crystals having a rhombic section of 60° or 120°. The figures 1 and 2, which we reproduce from a previous paper, may help to give a more definite form to these conceptions; but such representations are necessarily purely conventional symbols of conditions of which we have as yet no accurate knowledge, and to which, therefore, we can give no definite shape. The capability of such molecular macling as we have described may depend solely on the dimensions of the molecules; and in our figures we have represented such a condition, by giving to the sections of the molecules the form of ellipses of such dimensions that they can be inscribed in the rhomb of 60° and 120°. The conjugate diameters of this figure, when equal, subtend angles of 60° or 120°; and if the poles of the molecules are, as would be natural, at the ends of these lines, then, when the molecules were grouped as shown in Fig. 1, the unlike poles would fall directly over each other; so that the attractive and repulsive forces, centring at the poles, would hold the parts firmly together. The same molecules, if placed parallel to each other (as in Fig. 2), would be also in a stable condition, and the resulting rhombic section would have angles of 60° and 120°. On the other hand, although ellipses of other dimensions might be united as in Fig. 2, so as to give rhombic sections of every possible angle, yet only with ellipses of the dimension we have described, or those closely approaching this condition, would such a grouping be stable as is represented by Fig. 1. Of course, the molecules must have three dimensions; and, as before intimated, the ellipses are only conventional modes of expressing conceptions which are necessarily very incomplete. These symbols, however, will give form to our theory, and show why, among a series of isomorphous compounds crystallizing in the rhombic system, we might expect to find hexagonal crystals wherever among the various molecular

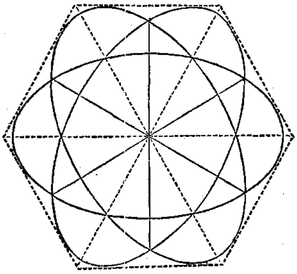


Fig. 1.

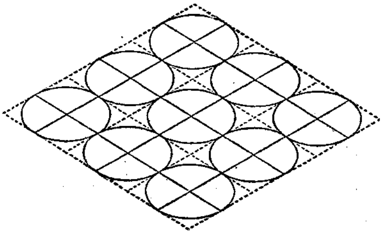


Fig. 2.

magnitudes the necessary dimensions were realized, although it is probable that there are other conditions which must also concur to produce this result.

Evidently, we have before us just such an isomorphous series as our theory anticipates, — a series of closely allied substances, in which the orthorhombic passes into the true hexagonal structure; and this furnishes us with an excellent opportunity for testing the theory we have advanced. If the crystalline molecules of the hexagonal iodide of antimony are really groups of three of the chemical molecules of this substance, then we might hope to find another condition of this substance in which the molecules were united, as in the crystals of the allied substances presenting rhombic forms with the angles of 60° or 120°; and, if such could be discovered, it would be reasonable to expect differences in the physical properties of the two isomers corresponding to the differences of structure. We were therefore led to search for a rhombic modification of the iodide of antimony, and with what success will soon appear. Before pursuing this subject, however, it will be best to describe some of the other physical properties of the hexagonal iodide.

The color of the hexagonal iodide of antimony is a brilliant vermilion red, which, however, in some preparations, is more or less tinged with yellow, in consequence of oxidation, and the formation of an oxiiodide on the surfaces of the crystals. The solution of the iodide of antimony in sulphide of carbon has a greenish yellow color, resembling that of uranium glass, and strikingly contrasting with the brilliant red color of the crystals which have been formed from it, — a fact whose significance will hereafter appear.

We made five determinations of the specific gravity of the hexagonal crystals of iodide of antimony; using for the purpose different preparations, and taking the specific gravity under petroleum which had been rectified over sodium, — the only liquid we could find that did not act on the substance. Even the rectified petroleum, however, acted slowly on this, as it did on the allied substances; but, during the time occupied in the determination, the effect was very slight, and no considerable error could have been thus caused. The following are the results of these determinations, all made at a temperature of about 24°C.: —

No.	Weight taken in grammes.	Sp. Gr. at 24°, referred to water at same temp.
1.	6.9730	4.807
2.	10.1393	4.895
3.	5.3506	4.812
4.	7.8868	4.893
5.	2.1517	4.833
Mean value		4.848

We made four observations of the melting point of the hexagonal iodide. For the first, we used a large amount of material, and, by dipping the bulb of the thermometer in the melted mass, observed the constant temperature while it was slowly solidifying. In the other experiments, we melted a few crystals in a glass tube, heated by a bath of sulphuric acid in the usual way. The result in the first determination was 167½°; in the last three, uniformly 167°, which is doubtless the true value. In the experiments with the tube, it was noticed that the point of solidification was about five degrees below the melting point.

Bringing together now the several results, we have the following comparison of the melting points of the three haloid compounds we have studied —

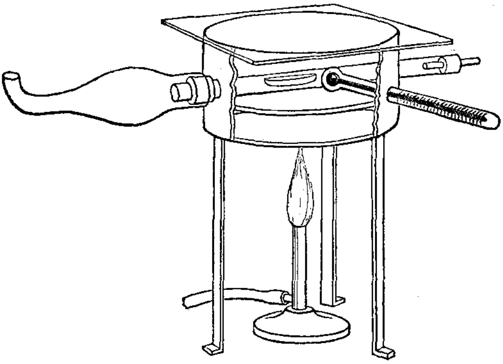
	Melting Point.	Differences.
Rhombic Antimonious Chloride	72°	
		21°
Rhombic Antimonious Bromide	93°	
		74°
Hexagonal Antimonious Iodide	167°	

From this it appears, that, instead of the equal differences we should naturally expect, the difference between the last two values is between three and four times as great as the difference between the first two. According to all analogies, the melting point of the normal iodide of antimony should be 114°; and the greatly higher value which we obtained is still another indication that the hexagonal iodide on which we experimented has an essentially different structure from that of the chloride or bromide of the same element with which it is here compared.

As yet, we have not been able to determine the boiling point of iodide of antimony with any accuracy. It is above, but apparently only just above, the boiling point of mercury; and, before the observation can be taken, it will be necessary to adapt some form of air thermometer to the necessary conditions.

ORTHORHOMBIC ANTIMONIOUS IODIDE.

We first met with this new substance while examining with a microscope the product obtained by subliming hexagonal iodide of antimony, in the apparatus represented in the accompanying figure, which we described in our previous paper (page 57); and we at once recognized, both by its color and its form, the isomeric modification of which we were in search. It appeared in small greenish yellow sprays, sparingly distributed among the red plates of the hexagonal iodide. Its color was precisely that of the solution of the iodide in carbonic disulphide; and, as shown in Plate II., the serrated edges of the crystalline sprigs presented very much the appearance of a picket fence, repeating at each point the characteristic angle of 60°.



It soon appeared that the yellow iodide was always formed when iodide of antimony was sublimed at a low temperature, and that this was the one condition necessary. Iodide of antimony begins to volatilize far below its melting point, even at 100° C.; and, if it is sublimed between two watch-glasses at a temperature not exceeding 114°, the yellow modification is the sole product. It condenses them in beautiful feather-like sprays, on which, however, no definite form can be distinguished. The process, also, is exceedingly slow, and the product very small. A much larger yield is obtained when the iodide is sublimed at a temperature just above its melting point, in a current of inert gas sufficiently strong to sweep the vapor at once into a cool condenser, — conditions, which are perfectly realized in the apparatus we have described.

The yellow iodide generally forms but a small portion of the product of the sublimation; but, by carefully regulating the temperature and gas current, the proportion can be largely increased, and it can then be picked out among the scales of red iodide with which alone it is mixed. To obtain, however, in this way, a sufficient quantity of the yellow iodide for analyses wholly free from its more abundant red associate, would have been a very tedious task; but this was not necessary, in order to establish the perfect isomerism of the two substances. Of the material used in the following analyses, more than nine-tenths consisted of the yellow iodide of antimony; and as the results agree so closely both with theory and with the parallel analyses of the associated red iodide, which could be easily and perfectly isolated, there can be no question whatever that the two have the same percentage composition. These analyses are reproduced from our previous paper, in order to bring together all the facts bearing on the present discussion.

COMPARATIVE ANALYSES OF RED AND YELLOW IODIDE OF ANTIMONY.

Red Hexagonal.			Yellow Orthorhombic.		
No.	Weight taken.	% of Iodine.	No.	Weight taken.	% of Iodine.
1.	1.1877	76.110	1.	0.4610	76.161
2.	2.3201	76.040	2.	0.3496	76.161
Mean value		76.075			76.161
Theory for SbI_3, when Sb = 120 and I = 127					76.047

As additional evidence of the isomerism of these remarkable substances, we may here, in anticipation of a fuller discussion of the subject, mention a fact which would be by itself conclusive. At a very moderate elevation of temperature, the yellow iodide of antimony is completely converted into the red modification, and under such conditions that there can be neither loss nor gain of material in the process.

In measuring the angles of the rhombic plates as they lay in a normal position under the microscope, we found that we could obtain the sharpest results by projecting the image on a sheet of paper by means of a camera lucida. We were then able to adjust a straight edge to one after the other of two edges of the crystal, and draw the corresponding lines, whose angular divergence we then measured with a protractor. These measurements gave, for the acute angle of the rhomb, as a constant result, 60°. The supplementary angles, 120°, are usually truncated; but not unfrequently (as shown on Plate III., Fig. 2) we meet with perfect terminations of this kind. When these occur on the same sprays as the acute angles, they uniformly appear in their proper relative positions, at right angles to the supplementary terminations, and we have frequently seen both terminations on the same rhombic plate. Less frequently, we find the sixty-degree angles truncated; and, as the result of such truncation, we have observed isolated hexagonal plates as perfect as those of the hexagonal iodide, from which outwardly they only differed in their color.

We have represented in Plate III., Fig. 1, a spray, presenting, for the most part, quite a different set of terminations from those before figured, which, although they are not quite so well defined as the others, nevertheless measure very constantly 82° and 98°. When found on the same sprays, the 98° terminations are, in general, parallel to the 60°, and the 82° to the 120°. Moreover, the 98° terminations are very frequently found bevelling the constantly recurring terminations of 60°, and, on the other hand, as often, the terminations of 120° bevel those of 82°.

Sometimes one of the two bevelling planes disappears, or, at least, is reduced to such small dimensions as to be imperceptible on the section as shown under the microscope. This kind of hemihedrism is shown, in the largest termination, both on Plates II. and III.

The angles 98° and 82°, or more accurately 98° 12′ 50″ and 81° 47′ 10″, are the angles of the rhombic prism $\{120\}$, derived from the prism $\{110\}$ of 120° and 60°. The relation of these two prisms is shown by the figure below the drawing, on Plate II.; and it will be noticed that all the termination edges in the drawing are parallel to one or the other of the lines whose relative positions are thus defined. It is a very remarkable property of the prism of 120°, that the derived prisms $\{120\}$ and $\{230\}$ have identically the same angles; the only difference being that the relative positions of these angles (98° 13′ and 81° 47′) are reversed. This is shown in the same figure as before, where the prism $\{230\}$ is also represented, but by dotted lines. The property referred to depends on the circumstance, that one-half and two-thirds of the tangent of 60° $\left(\frac{\sqrt{3}}{2} \text{ and } \frac{2}{\sqrt{3}}\right)$ are reciprocals, and must therefore be the tangent and cotangent of the same angle, which is 40° 53′ 35″, or one-half of the prismatic angle 81° 47′ 10″ named above. The same values are also equal, respectively, to the sine and cosecant of 60°, or to the cosine and secant of 30°. Not only, therefore, is the rhombic prism of 81° 47′ in either of the positions we have described, crystallographically compatible with the prism of 120°; but also the two diagonals of the first bear to each other precisely the same relation which the lateral axes of a direct hexagonal prism or rhombohedron bear to those of the corresponding inverse forms.

The prism 81° 47′ evidently corresponds to the prism of approximately the same dimensions, on the crystals of antimonious bromide $\{101\}$, Fig. 3, Plate I.; and this new condition of antimonious iodide is therefore closely isomorphous with the only known state of antimonious bromide and antimonious chloride.

The rhombic plates of the yellow iodide are quite uniformly marked parallel to the edges and sides of the rhomb, showing an evident tendency to the formation of domes and octahedrons, — a phenomenon so familiar in skeleton crystals.

Although the habit and grouping of the new crystals, the dimensions of their angles, and their relations to known forms, furnish very satisfactory evidence of their orthorhombic character, yet, from the nature of the case, this evidence is not demonstrative, and we were therefore desirous of obtaining the more conclusive evidence which optical characters give. The crystals, however, obtained as we have described, are usually so excessively thin that we were obliged to search a long time before we could find single plates sufficiently thick to give a distinct interference image. We did, however, at last obtain several plates which enabled us to observe all the important features of this instructive phenomenon. The hyperbolas were well marked, and separated by about six divisions of the scale of Groth's polariscope, which corresponds to an apparent angle between the optical axes of about 36°, although, on account of the thinness of the plate and consequent indefiniteness of the boundaries of the images, it was impossible to measure the angle exactly. The acute bisectrix was perpendicular to the faces of the plate, at least as closely as could be observed; the dispersion of the axes was very marked; and the hyperbolas were bordered with green on the concave and red on the convex side. Hence, $\varrho > \upsilon$. No differences could be seen in the coloration between the two ends or the two sides of the image. A very perfect isolated hexagonal plate enabled us to determine that the plane of the optical axes was parallel to the sides of the hexagon, as in most micas, and as represented in Fig. 4. This plate was not sufficiently

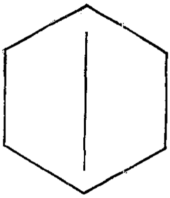


Fig. 4.

thick to show the hyperbolas distinctly; but, by combining it with a plate of a negative uniaxial crystal (tourmaline), we readily obtained the same familiar modification of the interference figure which is produced by a very thin plate of mica, and we were thus able, not only to determine that the character of the double refraction of our yellow plate was negative, but also to fix the position of the plane of the optical axes as just described.

We have stated that, at a very moderate temperature, the orthorhombic iodide changes to the hexagonal. It is now important to describe this remarkable phenomenon in detail. The change is not a gradual one; but suddenly, as soon as the required temperature is reached, a red spot appears, generally at one end of the rhombic plate, and then the red color rapidly spreads through the crystal, so that at any one point the change is instantaneous. Again, the change is attended with no disintegration of the crystal or loss of transparency; and not only the outline, but also all the minute markings, remain afterwards as sharp as before. Externally, there is simply a change of color from greenish yellow to bright red; and, by sudden cooling, it is perfectly easy to arrest the change so as to leave one part of the crystal red, while the rest remains yellow. The change, however, is attended with an entire alteration of structure; for the optically biaxial rhombic plate suddenly becomes absolutely uniaxial. Under the polarizing microscope, this change produces a very striking effect. In the dark field between crossed Nicols, the green rhombic plates show, of course, brilliant colors, whenever the diagonals of the rhomb lie obliquely to the planes of polarization; but, when the change takes place, a dark shadow suddenly spreads over each crystal, extinguishing this color, and then, on removing the analyzer, the very brilliant red color which the crystals have acquired appears. Under the polariscope, the sudden change from a biaxial to a perfectly uniaxial interference figure is still more striking. We have represented in Plate II. this remarkable phenomenon, as nearly as possible with a chromo-lithograph. The colors imitate very nearly, although not exactly, those of the natural crystals.

It becomes now a very interesting point to ascertain what is the exact temperature at which this singular change takes place. For this purpose, we heated on watch-glasses small quantities of the yellow crystals to different regulated temperatures, by means of a small air-bath; and in each experiment the temperature was maintained constant, at the selected point, for at least fifteen minutes. Thus we observed in one series of successive experiments: —

After heating to 120°, complete change.
” ” „ 107°, no change.
” ” „ 110° ” ”
” ” „ 111° ” ”
” ” „ 112° ” ”
” ” „ 114°, partial change.
” ” „ 114° complete „

These results were completely confirmed by similar experiments, all of which indicated that 114° is very closely the temperature at which the change first begins; and this result is in complete accordance with the fact we have before stated, that the red iodide of antimony, when sublimed below 114°, is completely converted into the yellow modification.

It is evident, from the above experiments, that the point of the change we have been discussing is fully as sharply marked as the melting point of a solid; and, by referring to the table of melting points on page 84, it will be seen that 114° (the temperature at which the change takes place) is the very point at which, theoretically, the normal iodide of antimony ought to melt. Evidently, then, the yellow orthorhombic iodide does undergo incipient fusion at this point; and the molecules, becoming thus free to move, regroup themselves, and the more stable structure of the red hexagonal iodide results.

Here, then, we have certainly a most remarkable confirmation of the theory we have advanced in regard to the molecular structure of hexagonal forms. The two isomers we have just described have enabled us to show that absolutely the same external form is compatible with the differences of structure which distinguish the orthorhombic from the hexagonal system; and this fact, only probable before, is now demonstrated. Secondly, the conditions under which one of these isomers changes into the other indicates clearly that the difference between the two substances is simply a difference of grouping of the same molecules, and also that in the red modification the molecules are more intimately united than in the yellow. When we attempt to go further, and explain what this more intimate grouping is, we of course soon enter the region of theory; but the analogy furnished by the superimposed mica plates is certainly very strong. We may now be said to know that the structure of an hexagonal crystal can be produced by a more intimate grouping of the molecules of an orthorhombic crystal, whenever the dimensions are such that the same external form is compatible with these two types of internal structure. We also know that the optical effect thus produced is like that obtained by superposing orthorhombic plates in a definite way. We further know, that, so long as these plates are kept of equal thickness, and their relative position maintained, the character of the effect is independent of their magnitude. Were the plates indefinitely thin and indefinitely small, there can be no question that a proportional effect would result, which, if indefinitely multiplied by passing the light through a great number of such superposed bundles, must give the same total effect as that obtained from a single bundle of thicker and larger plates. Now, molecules, in the sense in which we have used the term in connection with the theory we are discussing, are simply the elements of a certain definite orthorhombic structure, and have their analogues in small orthorhombic plates, — of mica, for example. By grouping these plates in a definite way, a certain optical effect is produced, without any change of external form. Precisely the same effect is the result of a change caused by heat in the mass of another orthorhombic material having the same crystalline form as the mica plates. Moreover, this change takes place at the exact temperature at which the parts of our protean material must acquire freedom of motion; and the obvious conclusion is, that in this change the elements of the orthorhombic crystal group themselves anew in the same way in which we group the mica plates in order to obtain a similar result. This is the outline of our argument. To enforce it, we have accessory facts, which show that the theory is in harmony with the accepted principles of polar forces and molecular mechanics, but these it is unnecessary to recapitulate. Of course, demonstration in such a case is out of the question; but we hope that we have been able to make clear that the theory of molecular macling, when stripped of the accessories which the conventional term "molecule" implies, is a close induction from the observed facts.

We have only one further point to make, before concluding the discussion of this portion of our subject. The color of the rhombic iodide of antimony is, as we have said, greenish yellow, recalling that of uranium glass; and, when either the yellow or the red iodide is dissolved in disulphide of carbon, the resulting solution has always the greenish yellow color of the first. In this solution, according to all theories, the molecules must be isolated; and the fact that this characteristic color is retained by the rhombic structure, and wholly changed in the hexagonal, is additional evidence that, while in the first the molecules act on the light as units, in the second their individual action must be modified by conditions depending on a peculiar mode of association.

MONOCLINIC OR CLINORHOMBIC ANTIMONIOUS IODIDE.

We have stated in the previous article, that, when a solution of antimonious iodide in carbonic disulphide is exposed to the atmospheric air under the influence of the solar light, the compound in solution undergoes a gradual oxidation, iodine is set free, and oxi-iodide of antimony is precipitated. If, after this action has continued for twelve or fourteen hours in the direct sunlight, the very dark colored solution is distilled over a water bath (at a temperature which, even at the end of the process, should not exceed the boiling point of the solvent), the greater part of the free iodine passes over with the distillate; and, by repeating the distillation several times after adding more of the solvent to the residue in the retort, almost the whole of the free iodine can be removed. When now the residue is again dissolved in the same solvent, and the solution is allowed to evaporate spontaneously, there is generally deposited, at first, a crop of the crystals of the red hexagonal iodide; but, on decanting the remaining solution, we obtained, as a final result, a small amount of yellow monoclinic crystals similar to the one figured on Plate I. Fig. 6. These represent a third condition of antimonious iodide, which, being more soluble in disulphide of carbon, are very easily separated from the hexagonal iodide with which they are associated.

The yield of monoclinic crystals in the process we have described is uncertain and irregular; and as yet we have been unable to bring the conditions of their formation wholly under command. After much experimenting, however, we have reached a few definite conclusions. To the crystals prepared as above, there frequently adhered a perceptible amount of free iodine, and we naturally questioned whether this impurity might not have some influence on their production; but after repeatedly recrystallizing the hexagonal iodide from solutions containing even a large excess of free iodine, and obtaining none of the monoclinic modification, we concluded that the iodine had no further effect than to modify the habit of the hexagonal crystals as already described on page 78. In like manner, finding in certain cases, when crude disulphide of carbon was used, that the yield of monoclinic crystals was unusually large, we at once suspected that the impurities originally existing in the solvent might be the determining cause of the change of conditions in the crystals deposited from it. We therefore experimented with some very strong-smelling disulphide of carbon which had become yellow by long exposure to light and air; but we could not obtain the monoclinic crystals, except in uncertain traces, until the disulphide with the dissolved iodide of antimony was exposed to the direct sun-rays for a long time, as above described. An equal amount of exposure before dissolving the iodide had no apparent effect. Again, after finding that iodide of antimony dissolved in disulphide of carbon is oxidized by ozonized air even in the dark, we sought to determine whether the monoclinic crystals were formed under these conditions independent of the action of the light; and, to this question also, our experiments returned a negative answer.

The production of the monoclinic crystals appears, therefore, to be independent of the oxidation of the iodide of antimony; although, like the latter, it is also due to the action of the sunlight. The effect seems to depend on a change produced in the solvent by the sun's rays, probably the same partial reduction of the carbonic disulphide which has been studied by O. Loew.* At least, it is certain that the monoclinic crystals contain a small amount of carbonaceous impurity, which they acquire from the solvent; and our theory is, that, under the action of the light, the crystalline molecules become as it were loaded by the adhesion of this material, and that the monoclinic habit is due to this circumstance. It may be mentioned, in this connection, that when the monoclinic crystals are redissolved in carbonic disulphide, and recrystallized, only a small part of the material, as a rule, retains the monoclinic form, the rest appearing in the more stable hexagonal condition. Still, there is a certain persistency of the new condition which would be natural with such a molecular adhesion as our theory assumes; and the remarkable adhesiveness of the material which is deposited by disulphide of carbon under the influence of the light is worthy of notice in this connection. Other facts bearing on this theory can be more intelligently discussed after this new form has been described.

The specific gravity of the monoclinic iodide of antimony was determined in the same way as that of the hexagonal modification (page 83), and with the following results: —

SPECIFIC GRAVITY OF MONOCLINIC ANTIMONIOUS IODIDE.

No.	Weight taken.	Sp. Gr. at 22°.
1.	1.2434	4.786
2.	2.2605	4.750
Mean value		4.768

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* Am. Journal of Science (2), xlvi. 363, and xli. 251.

It will be remembered that the mean value from five determinations of the specific gravity of the hexagonal modification was 4.848.

We made two analyses of the monoclinic iodide, each with a different preparation, and the iodine was determined as described in our previous paper. The small amount of impurity which would not dissolve in the tartaric acid solution was estimated in the first analysis, but not in the second, and we therefore give in each case the per cent of iodine in the crystals as they were weighed.

Analyses of Monoclinic Antimonious Iodide.

No	Weight taken.	Weight of Ag. I obtained.	% of Iodine.
1.	1.3087	1.8332	75.73
2.	1.8988	2.6605	75.70
Mean value			75.715
Theory when Sb = 120 and I = 127 . . .			76.047

The insoluble residue in the material of the first analysis weighed two and one-half milligrammes; and, were allowance made for this impurity, the corresponding percentage of iodine would be raised to 75.87. We found, however, that this material, like the hexagonal modification when crystallized from the same solvent, contained a trace of oxi-iodide of antimony, which of course must also be taken into the account before we can expect a close agreement with theory. So far as our analyses indicate, the total amount of impurity in the monoclinic crystals is not greater than that which we often find in the hexagonal crystals when crystallized, like the first, from a solution in carbonic disulphide. The difference seems to depend on the condition in which the impurity is present: and the conception we have formed is, that in the first case the impurity is a mere admixture like dirt entangled by the crystalline structure, while in the last case it actually adheres to the molecules. This molecular adhesion is induced by the action of the light; and our theory assumes, that, while the primitive molecules thus loaded are prevented from macling, so as to produce the triads of the hexagonal structure as shown in Fig. 1, so, also, these loaded molecules, when united in the normal way as shown in Fig. 2, form a crystal differing essentially, although still not very widely, from the now familiar orthorhombic prism of sixty and one hundred and twenty degrees.

The perfect isomerism of the monoclinic crystals with the other forms of antimonious iodide is shown by the readiness with which they change into the hexagonal iodide even more strikingly than by the above analyses. The general color of these crystals, like that of the orthorhomic crystals, is yellow; but the shade of color varies considerably under different circumstances, depending obviously on the impurity present. The purest crystals we have obtained had the same green tinge as the orthorhombic plates represented by Plate II., although less pronounced; and from this the tint varied through lemon yellow to quite dark brown, the brownish tinge evidently resulting from the free iodine, which, as we have said, frequently adheres to the crystals. Now, when heated, the monoclinic crystals (like the orthorhombic plates) acquire a red color and hexagonal structure at a temperature below that at which they melt. But the point of the change is by no means so definite as in the former case; and we observed circumstances connected with it obviously depending on the impurities present, which have an important bearing on the theory we have advanced above.

In the first place, we found that the yellow monoclinic crystals would bear a temperature of from 120° to 125°, without undergoing change. The experiments were conducted like those with the orthorhombic crystals already described (page 90); and comparative experiments were made in which the two modifications were heated side by side. The orthorhombic crystals were uniformly converted into the red modification at 114°, while the monoclinic were only slowly altered even at 125°. Of one experiment, we have the note, Some yellow crystals remain after heating for half an hour at 125°; and of another, The change is gradual: the color deepens; the crystals become opaque, and soon after melt.

On the other hand, we observed that brownish crystals, which had been deposited from a solution containing a large excess of iodine, often reddened at the boiling point of water. In one experiment, in order to obtain a direct comparison, we heated for one hour side by side in a steam bath, portions of the following three different preparations: —

1. Greenish yellow orthorhombic plates obtained by sublimation.
2. Lemon yellow monoclinic crystals similar to the one figured on Plate I. Fig. 6.
3. Brownish monoclinic crystals evidently occluding free iodine.

Except the loss of evaporation, both 1 and 2 underwent no change, but 3 was wholly converted into the red modification. The change was attended with some efflorescence, and the crystals generally became more or less opaque; but sometimes they retained their transparency quite perfectly, and we noticed instances in which the change spread through these crystals as through the orthorhombic plates before described. It is evidently the same change in both cases; and the best explanation we can give of the phenomena is, that, while the fixed carbonaceous occlusion, by loading the molecules, renders them less mobile, and thus tends to prevent the change, the volatile iodine, on the other hand, in breaking away from its attachment, destroys the unstable equilibrium of the molecules, and thus induces the change at a lower temperature than it would otherwise take place.

The habit of the clinorhombic or monoclinic crystals of antimonious iodide varies considerably under different conditions between prismatic and tabular forms. As in the case of the hexagonal iodide, the presence of an excess of iodine in the solution seems to determine the production of elongated prisms, while, after the solution has been freed from iodine, more tabular crystals are deposited. We measured numerous angles on crystals of these different types; and although the forms were obviously the same, and the similar angles very nearly equal, yet we observed differences in these angles amounting to fully half a degree, even when the several measurements must have been accurate to a minute. Such a variation in the angles was to be expected under the circumstances, and is wholly in harmony with the theory we have formed of the structure of these crystals. The crystal figured on Plate I. Fig. 6, and whose dimensions are given below, was selected from the product of the most successful crystallization we have yet made. On it were united all the forms we have observed on any of the crystals of this modification of antimonious iodide. It had a lemon yellow color, and was obviously of the tabular type. The crystal was very perfect, and most of the angles admitted of accurate measurement. The results are given below. They were verified by numerous repetitions of the measurements in zones and by comparison with similar angles measured on other crystals of the same preparation. As usual, we deduced the elements of the crystal from the measured values of three selected angles; and it will be noticed that the measurements of the other angles agree very closely with the values calculated from these data assumed to be fundamental. Indeed, we have seldom obtained better results with artificial crystals.

CRYSTALLINE FORM OF ANTIMONIOUS IODIDE.

MONOCLINIC VARIETY.

Forms {001}, {110}, {210}, {011}, {211}

Fig. 6, Plate I.

FUNDAMENTAL ANGLES.

Between normals of	210	and	$\bar{2}10$	measured	75° 21′
„ „ „	001	„	210	„	105° 30′
„ „ „	001	„	211	„	53° 50′

Angle of Axes 70° 16′
Clinodiagonal, á = 1.6408
Orthodiagonal, b = 1.
Vertical, ć = 0.6682

				Measured.	Calculated.
Between normals of	100	and	210	37° 40½′	37° 40½′
„ „ „	210	„	110	19° 24′	19° 24′
„ „ „	110	„	$\bar{1}10$	65° 51′	65° 51′
„ „ „	001	„	110	100° 36′	100° 34½′
„ „ „	211	„	$2\bar{1}1$	61° 37′	61° 36′
„ „ „	011	„	$01\bar{1}$		115° 40′

Angle between basal edges made by	{001}	and	{210}	100° 16′
„ „ „ „ „	{001}	„	{110}	62° 44′

Basal edge angle at end of clinodiagonal	100° 16′	corresponds to the prismatic angle	104° 39′
Do.	62° 44′	do.	65° 51′

Vertical axis on the assumption that the forms {001} and {211} are orthorhombic 0.867

The crystals have a very well defined cleavage, parallel to the basal plane.

We endeavored to supplement the above crystallographic measurements by a study of the optical characters of the crystals; but, unfortunately, in their natural state, these crystals were not well adapted to the purpose, and the best we have hitherto obtained have been too small, and the material too easily altered, to admit of the preparation of sections. We shall return to this work at the first opportunity, and we hope for better success in the future. For the present, we can only say that the axes of elasticity in the plane of symmetry make a small angle with the basal plane and its normal. With a minute and imperfect section, the rough measures which were alone possible under the microscope gave for the value of this angle about 8°. And, further, that we were unable to see the interference figure through the basal planes under conditions which were far more favorable than those under which the hyperbolas were seen through the orthorhombic plates: for the tabular monoclinic crystals we used in these observations were both larger and thicker than the plates; and, in order to secure an accurate comparison we examined the different crystals in succession. We also examined several cleavage sections parallel to the basal planes, with no better success; probably because the angle between the optical axes is sufficiently great to determine total reflection. The high index of refraction of the substance would naturally produce this effect at a very moderate value of the optical angle, and unfortunately the crystals at once lost their transparency when immersed in the liquids generally used in such cases.

The above results, although partial, are important as indicating a marked difference between the two classes of crystals we have last studied. Nevertheless, a comparison of measurements will show that the dimensions of the *basal section* of the monoclinic crystals differ by only two or three degrees from those of the orthorhombic plates before described. Thus, we have 62° 44′ in place of 60°, and 100° 16′ in place of 98° 13′; and if, from the angle of the hemi-octahedral plane on the basal section, we calculate the length of an assumed rectangular vertical axis, the value obtained (0.867) does not differ very greatly from the values of the corresponding axis for the chloride, bromide, and hexagonal iodide of antimony, as given on page 113. Moreover, on recrystallizing the monoclinic iodide from a solution in pure sulphide of carbon, we have in two instances obtained — mixed with the hexagonal iodide, which is always the chief product — microscopic rhombic plates showing all the planes of the octahedron, and whose angles (as nearly as they could be measured under the microscope) were 60° and 120°.

As it seems to us, the natural inference from the facts we have developed is the theory already intimated. According to this theory, the monoclinic differs chemically from the orthorhombic iodide, only in containing a small amount of impurity. The molecules are supposed to be similarly constituted and similarly grouped together; but, by the adhesion of the impurity to the molecules, a certain difference of form results. This difference, although usually regarded as fundamental (for it is the difference between two crystalline systems), does not appear, as thus viewed, either great or essential; and the fact that similar differences of form are also met with in the mineral kingdom, among the micas, the vermiculites, and the chlorites, without the corresponding differences of chemical composition and physical qualities which an essentially different crystalline structure would imply, tends to sustain our theory. The subject is obviously one of great importance, and we may hope that the further study of these artificial products may serve to elucidate what has been a very obscure chapter in the science of mineralogy. We propose to continue the investigation, and as soon as larger and better crystals of the several conditions of antimonious iodide can be prepared, we shall repeat and complete the optical measurements. Meanwhile, we are studying the allied iodides, which promise further results. Every one who has had experience with this kind of work knows how easily an observer may fall into error by mistaking in these optical phenomena the delicate shades or features on which important distinctions frequently depend, and this is especially true when, as in the present case, the conditions are not the most favorable. An examination of larger and more perfect crystals will undoubtedly correct some of our data, and may modify some of our conclusions. The results here given are the best that could be obtained with the material at our command, and must be regarded as provisional until better material can be secured.

Tschermak maintains that the Muscovite micas are monoclinic crystals, of which the acute bisectrix makes a very small angle with the plane of cleavage; and it is possible that the crystals of antimonious chloride, bromide, and iodide, which we have studied, should partake of a similar structure; and that the thin, rhombic plates of antimonious iodide obtained by sublimation should differ from the monoclinic crystals of the same compound only in their habit. At least, with our present imperfect measurements, we cannot disprove such a theory; although the not necessarily incompatible theory advanced above seems to us the more probable of the two, and the only one which is consistent with the facts as they at present appear. Yet, if the other view should prove to be the more correct, the general result of the discussion in this paper would not be affected: only we must extend also to such monoclinic crystals as we have described the principles here illustrated in regard to the relations of hexagonal forms.

The chlorites, the vermiculites, and the micas, whose crystallographic relations first suggested to us the theory of molecular macling, which the new facts developed in this paper have so fully confirmed, are all foliated minerals, of whose crystals the optical axis or acute bisectrix is either normal, or inclined at only a small angle to the plane of easy cleavage. With the crystals of antimonious iodide, both hexagonal and monoclinic, there is also an easy cleavage, parallel to the basal plane; and there is also a similar, if not an identical, relation of the optical axes. There is, however, no other evidence of a foliated structure, nor any sign of interlamination, such as we observed in those minerals. The crystals appear to be perfectly homogeneous, and (saving their great brittleness) cleave more like crystals of topaz than those of mica. The difference between the effect of interlamination and that which, as we suppose, results from molecular macling, must not be overlooked, although the optical phenomena in the two cases are so similar. What we called, in our paper on the vermiculites, interlaminar macling does not involve any essential change in the substance of the mineral; but molecular macling produces a new, although isomeric, substance. The red and the yellow antimonious iodides are as different substances as calcite and arragonite; and, as we conceive, the difference in the two cases is of the same kind.

The facts developed in this paper all point to a more intimate relation between the different crystalline systems than has generally been supposed to exist, and are in complete harmony with the opinion we have frequently expressed, — that differences of crystalline system are not necessarily more fundamental than corresponding differences of dimension in the same system.

ANTIMONIOUS OXI-IODIDES ($SbOI$ and $Sb_4O_5I_2$).

We have already, page 92, described the very remarkable chemical reaction which takes place when a solution of antimonious iodide in carbonic disulphide is exposed to the action of light and air. The reaction is chiefly that expressed by the formula, —

$$SbI_3 + O = SbOI + I\text{-}I;$$

but this is, to a very limited extent, accompanied by the more complex reaction, —

$$4\ SbI_3 + O_5 = Sb_4O_5I_2 + 5\ I\text{-}I.$$

The oxi-iodides of antimony, being insoluble in carbonic disulphide, are precipitated as an amorphous yellow powder, while the free iodine remaining in the solution changes its original greenish yellow color to a deep red, so deep that it soon appears black by reflected light. The change of color in the direct sunlight is very rapid, and forms a most striking phenomenon, which can readily be shown on the lecture table. When the direct rays of the sun are not available, the reaction can be produced by burning a few feet of magnesium ribbon. It is by far the most striking example of oxidation by the sun's light which has yet been discovered; and may, therefore, as a lecture experiment, be brought in striking contrast with the reduction of argentic chloride by the same agent, — a change which it rivals in extent, if not in rapidity. It has been maintained * that while the more refrangible rays of the solar spectrum exert a reducing action on metallic compounds, both binaries and salts, the less refrangible rays (the yellow as well as the red) produce the contrary effect, and tend to increase the oxidizing action of the atmosphere. In the phenomenon we are studying, the oxidation is actually determined by the sun's light, and in the most marked manner; and, as this is the first definite example of such action which has been observed, it became a very interesting question to inquire, in what part of the solar spectrum the action was most intense. We therefore exposed the solution, in test-tubes, to the sun's rays at different parts of the solar spectrum, but under otherwise identical conditions; taking care, of course, to protect the tubes from any other radiation. The spectrum was projected in the usual way, with a lense and prism of glass; and we found that, while the brilliant red and yellow rays caused no sensible change of color, the comparatively faint blue and violet rays produced a very marked effect. Our method of experimenting was not delicate enough to show the precise point of maximum effect; but it was evident that the whole order of the phenomena was the same as in the case of argentic chloride and similar photographic preparations.

As we have before stated, the solution of iodide of antimony undergoes no change in contact with ordinary air, so long as it is kept in the dark; and since the amount of iodine set free under the influence of the light can be readily estimated, and since this is the measure of the chemical action, it is evident that the new reaction affords a direct

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* Étude sur la Part de la Lumière dans les Actions Chimiques, et en particulier dans les Oxydations par M. P. Chastaing. Annales de Chimie et de Physique (5), XI. 145. June, 1877.

means of measuring the amount of chemical change caused by solar radiation.

We have also stated, that, even in the dark, iodide of antimony is oxidized by ozone. The experiment is easily made by passing through the disulphide of carbon solution a current of air which has been ozonized by electricity. The action is very marked, but not so rapid as in the direct sunlight. The products are the same in both cases, — oxi-iodide of antimony and free iodine; and, under the influence of the sun's direct rays, all the iodide of antimony can be thus, with time, removed from the solution.

The reaction we are considering was first observed in a closed flask, and the circumstances were such that we did not at first suspect the important part which the atmospheric air played in the process. This however, became evident as soon as we had examined the products of the reaction; and we then made experiments to determine whether any reaction would take place out of contact with the air. For this purpose, we sealed up the solution in flasks from which the air had been displaced by carbonic dioxide, and, under these conditions, exposed the solution to the direct sun's light. But, although it was easy in this way to preclude any considerable change, we did not succeed in preventing it altogether. A slight reddening and turbidity indicated at least the beginning of oxidation, and this we traced to the oxygen gas held in solution by carbonic disulphide. But, after this small amount was exhausted, the action was wholly arrested, though we exposed the solution for days to the brightest sunlight.

Carbonic disulphide obviously aids the reaction by dissolving oxygen gas, as well as antimonious iodide; but we must not overlook what we stated in the previous paper, — that the same oxidation may take place independently of this solvent. The crystals of antimonious iodide, the yellow as well as the red, soon become coated with oxi-iodide, when exposed to the light and air; and we have lost a number of fine specimens from this cause. Under the microscope, the oxi-iodide appears as a yellow efflorescence, which soon destroys the transparency of the mass; and the odor of free iodine can be perceived on opening the bottle in which the preparation has been kept. This action, of course, is comparatively slow, but not less definite than that which we have been previously considering.

The oxi-iodide which, during oxidation, falls from the solution of antimonious iodide, is an amorphous yellow or brownish yellow powder. As it is insoluble in carbonic disulphide, it can easily be separated and cleaned by filtration and washing with this liquid. We made analyses of three different preparations, with the following results: —

ANALYSES OF ANTIMONIOUS OXI-IODIDE.

No.	Weight taken.	Weight of AgI obtained.	% of Iodine.
1.	0.1105	0.0837	40.94
2.	0.5728	0.4511	42.56
3.	0.2411	0.2059	46.15
		Theory for SbOI	48.29
		„ „ $Sb_4O_5I_2$	31.20

Whence it is evident that the material consists chiefly of SbOI; and, by regulating the action, this substance can be thus obtained in a nearly pure condition, as the last analysis shows.

The chemical constitution of the precipitated oxi-iodide is also very plainly indicated by the successive changes which it undergoes when heated in an atmosphere of inert gas, especially if they are studied in connection with the precisely similar changes which the well crystallized, and therefore more definite, oxichloride of antimony (SbOCl) undergoes under the same conditions as already described, page 63, of this volume. We experimented on the oxi-iodide with the apparatus also described and figured in our previous paper; and we found that, from the dried precipitate, when heated in a current of carbonic dioxide gas, antimonious iodide begins to sublime at 150°. At 200°, the sublimation became active, and continued until a definite amount of SbI_3 has been driven off, when it wholly ceased. In external appearance, the residue differed only slightly from the original substance; but when analyzed it gave the following result:—

Analysis of residue after heating at 200°, until sublimation ceased.

Weight taken.	Weight of AgI obtained.	% of Iodine.
1.8853	1.1512	32.99

On now heating the residue more intensely, it underwent no further change until the temperature rose above 350°; but, at a low red heat, antimonious iodide again sublimed, and there was left, as the final residue, beautifully crystallized antimonious oxide — both orthorhombic prisms and octahedrons. The reactions were obviously these:—

$$\text{At } 200°,\ 5\ SbOI = Sb_4O_5I_2 + SbI_3.$$

$$\text{At low red heat, } 3\ Sb_4O_5I_2 = 5\ Sb_2O_3 + 2\ SbI_3.$$

On examining the residue of the last reaction, before the change was complete, we have observed, mixed with the very brilliant colorless crystals of antimonious oxide, faintly yellow crystals, which had a well-marked monoclinic form, resembling that of the crystals of antimonious oxichloride ($Sb_4O_5Cl_2$); which are deposited from aqueous solutions of antimonious chloride which contains a deficiency of tartaric acid, and which we shall describe hereafter. These crystals were only microscopic objects, and far too small both in size and quantity for chemical analysis or crystallographic measurement. We were, however, able to prove, both that they contained iodine, and that they were converted into Sb_2O_3, on further heating; and there can be, therefore, no question that they were crystallized, $Sb_4O_5I_2$.

ANTIMONIOUS OXIBROMIDES (SbOBr and $Sb_4O_5Br_2$).

Under the influence of the air and the direct sunlight, a solution of antimonious bromide undergoes a slow oxidation like that we have just studied, but to a far less extent. Bromine is set free, and an oxibromide of antimony is deposited in an amorphous brownish powder; but the action is so slight, that, even after several weeks' exposure, we were unable to obtain, from a considerable volume of the solution, a sufficient amount of the precipitate for analysis. We were only able to prove that, like SbOI, it is decomposed when heated in two stages, leaving a residue of Sb_2O_3; but this reaction was sufficient to indicate that it consisted mainly, at least, of SbOBr. The precipitate contained a considerable amount of carbonaceous material, also separated from the solvent by the light; and it evidently owed its color to this impurity. Pure SbOBr would undoubtedly be colorless.

The compound $Sb_4O_5Br_2$ can readily be obtained, by heating in a sealed tube, to a temperature of 160°, a mixture of antimonious bromide and absolute alcohol, according to the method employed by Schaeffer* for preparing the oxichlorides. Some beautifully crystalline oxibromide has been recently prepared in this way by Mr. Clifford Richardson, a student of this laboratory. The crystals were distinctly monoclinic, although too small for measurement. Mr. Richardson's analysis gave the following result: —

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* Berichte der Deutschen Chem. Gesell. 1868.

ANALYSIS OF ANTIMONIOUS OXIBROMIDE.

PREPARED WITH ALCOHOL.

Weight taken.	Weight of AgBr obtained.	% of Bromine.
0.6634	0.3424	21.96
Theory for $Sb_4O_5Br_2$ when Sb = 120		22.22

Several attempts were made to prepare SbOBr, by using a larger proportion of antimonious bromide as compared with the alcohol, according to the indications furnished by Schaeffer's experiments with antimonious chloride; but the product was uniformly $Sb_4O_5Br_2$; nor have we been, as yet, more successful in isolating the compound in other ways.* An analysis of the white amorphous precipitate formed by the action of water on antimonious bromide showed that it also consisted essentially of the same; $Sb_4O_5Br_2$. The following result was obtained by Mr. Richardson: —

ANALYSIS OF ANTIMONIOUS OXIBROMIDE.

PRECIPITATED BY WATER.

Weight taken.	Weight of AgBr obtained.	% of Bromide.
0.2281 gramme.	0.1155	21.54

ANTIMONIOUS OXICHLORIDES (SbOCl and $Sb_4O_5Cl_2$).

These two compounds were prepared in a crystalline condition by following the directions given by Schaeffer in the paper already referred to. The crystals of SbOCl were not described by Schaeffer. Those obtained by Mr. Richardson were from half a millimetre to a millimetre in length, and enabled us to determine their crystallographic dimensions with approximate accuracy. They were evidently monoclinic, and presented the planes of an oblique rhombic prism with a klinodome and pinacoids.

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* We also tried to prepare crystallized oxi-iodides of antimony by Schaeffer's method, but without success. A solution of antimonious iodide in absolute alcohol yields, without heating, an abundant precipitate of oxi-iodide, but as a perfectly amorphous powder. The material was evidently a mixture of the two oxi-iodides we have distinguished. Analysis gave for one preparation 46%, and for another 40.58%, of iodine.

Crystalline Form of Antimonious Oxichloride (SbOCl).

Prepared by Schaeffer's Method.

Monoclinic System.

Forms {110}, {011}, {001}

Fig. 7, Plate I.

Angles Measured.

Between normals, —

110 on $\bar{1}10$	98° 2′
011 „ $01\bar{1}$	107° 14′
110 „ 001	100° 8′
001 „ $\bar{1}\bar{1}0$	79° 52′

From these we calculated : —

Clinodiagonal, $\acute{a}$ = 0.8936
Orthodiagonal, b = 1
Vertical, $\acute{c}$ = 0.7587
Angle of Axes, = 76° 31′

By referring to page 97, it will be seen that these crystals are closely isomorphous with those of the monoclinic, antimonious iodide. We examined them also with a polarizing microscope, and found that, when the light passed normally to the prismatic faces, the principal optical sections made angles of 40° and 50°, respectively, with the prismatic edges.

The following analyses were made by Mr. Richardson. In the first three, the oxichloride was decomposed by boiling over it a solution of pure sodic carbonate. In the last, it was dissolved in a concentrated aqueous solution of tartaric acid.

Analyses of SbOCl.

Prepared by Schaeffer's Process.

No.	Weight taken.	Weight of AgCl obtained.	% of Chlorine.
1.	0.5055 gram.	0.4158 gram.	20.35
2.	0.7268 „	0.5997 „	20.41
3.	0.8367 „	0.6915 „	20.45
4.	0.5476 „	0.4488 „	20.28
	Mean value		20.37
	Theory Sb = 120		20.70

The want of closer agreement in the results, both with each other and with theory, we traced to a slight admixture of $Sb_4O_5Cl_2$; and we found that this last compound was by far the more readily formed of the two, and, except in the single experiment by which the preparation subsequently analyzed was obtained, the chief product of the reaction was largely mixed with the more oxygenated compound, even when the prescribed formula had been closely followed.

By operating with several hundred grammes of antimonious chloride and alcohol, we obtained the compound $Sb_4O_5Cl_2$ in beautiful large crystals, some of which were over a centimetre in length. We used for the purpose a cylindrical vessel of platinum, having a capacity of about 300 cubic centimetres; which, when covered with a lid of the same material, fitted exactly the interior of a Papin's digester, made nearly after the pattern recommended by Frankland.* This device was suggested by the "soda-water" fountains described on page 118 of the previous volume of these "Proceedings." The outside shell of such fountains must necessarily be very strong, and is now often made of steel plates; but the aerated water is held by an interior vessel, which, though independent of the shell, forms its lining. This inner vessel may be even of glass, for it bears no strain; since a small aperture through the neck equalizes the pressure on the outer and inner surfaces, and the "lining" fits the shell so tightly that no space is left into which the contents can overflow.

The general form of the crystals of $Sb_4O_5Cl_2$, prepared as we have described, is shown by Fig. 8, Plate I., and they are evidently more highly modified than those figured by Schaeffer. They are frequently terminated at the two ends, and usually differently terminated, as our drawing represents. At one end, we have an acute tetrahedral solid angle, formed by the meeting of the planes of a hemioctahedron with those of a vertical dome, while at the other end we have an edge formed by the meeting of the single basal plane with the single plane of an orthodome found on the crystal. These crystals thus present a very striking example of hemihedrism at the terminations, and we propose to examine hereafter their polar relations. The faces have a high vitreous lustre, and many of the angles can be measured with great precision. Unfortunately, however, but as a natural result of the multiplication of the octahedral planes, the faces of the principal prism are generally striated parallel to the basal intersections, and this striation renders more or less uncertain the measurements of the angles between the octahedral and the prismatic faces, on which we had chiefly to depend for determining the position of the vertical axis. The value of the angle $\bar{1}12$ on $\bar{1}10$, which we selected as one of the fundamental data, was obtained by comparing a number of crystals on which the condition of the faces was especially favorable, and is the most probable value deduced from many observations. Nevertheless, this is the one doubtful element, and may be in error to the extent of a few minutes. The result was checked by measuring the angle of $\bar{1}12$ on 110, which, although not favorably situated as a measure of the fundamental dimensions of the crystal, was useful as a proof of the accuracy of the work by which they were deduced; and the table below shows how well it bears this test. As in this last measurement, the reflected image of the signal crossed the striations at nearly 90°, the signal was comparatively well defined; and the same was true in the measurements of the prismatic angles, which agreed very closely on different crystals. The hemioctahedral planes $\{\bar{1}12\}$ were by far the best developed of all the planes of this class; but we selected in preference, as the fundamental octahedron, a subordinate form $\{\bar{1}11\}$, to which the associated planes bore a simpler numerical relation. The planes $\{\bar{1}13\}$ were so small that the reflection of the signal was seen only by flashes. The planes $\{\bar{3}31\}$ were the best defined of an indefinite zone between $\{\bar{1}11\}$ and $\{\bar{1}10\}$; among which others, with still higher parameters, might have been doubtfully distinguished. We have previously called attention to similar zones of planes* with high but yet definite numerical ratios. They are by no means an exceptional phenomenon, and their crystallographic interpretation seems to us worthy of more attention than it has received. The result of our measurement is given in the following table:—

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* Watt's Dictionary of Chemistry, article *Bath,* i. 520.

CRYSTALLINE FORM OF ANTIMONIOUS OXICHLORIDE ($Sb_4O_5Cl_2$).

PREPARED BY SCHAEFFER'S METHOD.

MONOCLINIC SYSTEM.

Forms $\{\bar{1}10\}$, $\{\bar{1}11\}$, $\{\bar{1}12\}$, $\{\bar{1}13\}$, $\{\bar{3}31\}$, σ $\{101\}$, σ $\{100\}$

Fig. 8, Plate I.

FUNDAMENTAL ANGLES.

Between planes	$\bar{1}\bar{1}0$	and	$\bar{1}10$	86° 49′
„ „	$\bar{1}12$	„	$\bar{1}10$	156° 42′
„ „	$\bar{1}\bar{1}2$	„	$\bar{1}12$	112° 7′

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* These Proceedings, vol. iii., page 87.

From these, we calculated —

Clinodiagonal, $\acute{a} = 1.239$
Orthodiagonal, $b = 1.$
Vertical Axis, $c = 3.082$
Angle of Axes, $C = 58° 38'$

ANGLES ARRANGED IN SERIES.

				Calculated.	Observed.
Between normals	$00\bar{1}$	and	$11\bar{2}$	45° 45′	45° 47′
„ „	$00\bar{1}$	„	$11\bar{1}$	55° 50′	55° 53′
„ „	$00\bar{1}$	„	$33\bar{1}$	64° 17′	
„ „	$00\bar{1}$	„	110	69° 3′	68° 50′
„ „	$00\bar{1}$	„	$\bar{1}0\bar{1}$	97° 54′	97° 33′
„ „	110	„	$\bar{1}10$	86° 49′	86° 49′
„ „	110	„	$\bar{3}31$	88° 47′	88° 38′
„ „	110	„	$\bar{1}11$	92° 12′	
„ „	110	„	$\bar{1}12$	96° 11′	96° 14′
„ „	110	„	$\bar{1}13$	99° 12′	99° 2′
„ „	$\bar{1}\bar{1}3$	„	$\bar{1}13$	57° 30′	
„ „	$\bar{1}\bar{1}2$	„	$\bar{1}12$	67° 53′	67° 53′
„ „	$\bar{1}\bar{1}1$	„	$\bar{1}11$	80° 9′	
„ „	$\bar{3}\bar{3}1$	„	$\bar{3}31$	89° 2′	
„ „	$\bar{1}\bar{1}0$	„	$\bar{1}10$	93° 11′	93° 11′
„ „	$\bar{1}10$	„	$\bar{3}31$	4° 46′	
„ „	$\bar{1}10$	„	$\bar{1}11$	13° 13′	
„ „	$\bar{1}10$	„	$\bar{1}12$	23° 18′	23° 18′
„ „	$\bar{1}10$	„	$\bar{1}13$	30° 52′	

We have not been able to study in detail the optical characters of these crystals; but observations with the polarizing microscope indicate, that, when resting on their prismatic planes, one of the principal optical sections is approximately, but not quite, parallel to the prismatic edges, making an angle with it of about 5°.

An analysis of the above crystals, made by Mr. Richardson, gave the following results: —

ANALYSIS OF ANTIMONIOUS OXICHLORIDE ($Sb_4O_5Cl_2$).

PREPARED BY SCHAEFFER'S PROCESS.

Weight taken.	Weight of AgCl obtained.	% of Chlorine.
0.6376	0.2879	11.17
Theory for $Sb_4O_5Cl_2$ when $Sb = 120$		11.25

We have already stated that this same compound is deposited from aqueous solutions of antimonious chloride containing less than a definite proportion of tartaric acid, under conditions which are given at length in our previous paper (page 23, of this volume). The crystals thus obtained differ wholly in general aspect from those we have just described. While the former were acicular, these are tabular, and, instead of being isolated, generally group themselves in tufts; which, although sometimes a millimetre in diameter, consist of crystals so small and so closely compacted together, that hitherto we have found it impracticable to separate and measure them. As seen under the microscope, the crystals appear distinctly monoclinic; the tufts presenting terminations similar to those of epidote, and the crystals showing the same tendency to growth in the direction of the orthodiagonal which is so characteristic of that mineral species; while, at the same time, the packing together of the tabular crystals in tufts recalls the phenomenon so familiar on specimens of calamine and prehnite. Assuming that the terminal planes at the ends of the orthodiagonal are the planes of a vertical prism, and the plane of twining the basal plane, then such rough estimates of the axial inclination as we have been able to make with the microscope would indicate that these crystals are much less oblique than the last, and more closely allied in form to the crystals of SbOCl before described. Like these, they frequently present the planes of a klinodome $\{011\}$, Fig. 7, Plate I., which never appear on the other type of crystals (Fig. 8), and indeed would hardly be compatible with it. By regulating more carefully the amount of tartaric acid in the solution, we hope to obtain hereafter larger crystals of this last variety of $Sb_4O_5Cl_2$, whose exact measurement will settle the question in regard to the relation of the two forms.

We analyzed with great care the crystals of $Sb_4O_5Cl_2$ deposited by the tartaric acid solution, in order to obtain additional evidence in regard to the atomic weight of antimony. By the methods already described, both the antimony and the chlorine were determined; while the oxygen was estimated, as is usual, by loss. The following are our results: —

ANALYSIS OF ANTIMONIOUS OXICHLORIDE ($Sb_4O_5Cl_2$).

CRYSTALLIZED FROM A TARTARIC-ACID SOLUTION.

	Found.	Theory.
Antimony . .	76.10	76.06
Chlorine . .	11.22	11.25
Oxygen . . .	12.68	12.68
	100.	100.

We also determined the specific gravity of these same crystals; which we found to be, at the ordinary temperature, 5.014.

OXICHLORIDE OF ANTIMONY ($Sb_8O_{11}Cl_2$).

As is well known, precipitated oxichloride of antimony (powder of Algaroth), when washed with hot water, undergoes a gradual decomposition; yielding after long-continued washing pure antimonious oxide, and hydrochloric acid, which is removed by the water. It is also known that, if the snow-white bulky precipitate is left under the liquid for a few days, it forms a grayish white mass, consisting of brilliant microscopic crystals, which are described by Johnston and Miller as oblique, rectangular prisms, having the obtuse summits replaced by planes. The amorphous precipitate is undoubtedly a mixture of the two compounds we have just studied, in varying proportions, depending on the conditions of the precipitation; but the crystals are evidently $Sb_4O_5Cl_2$. Johnston's analysis gives the exact theoretical per cent of chlorine (11.25); and the mean of two analyses by Peligot gives the same.* We can find no satisfactory evidence of a definite compound between SbOCl and $Sb_4O_5Cl_2$; and the fact that, when gradually heated, SbOCl manifests but one stage in its decomposition, seems to indicate that such a compound cannot exist. We have carefully studied this decomposition, and we would refer to the description of the phenomena which we gave in our previous paper, page 63, of this volume.† There was more probability that a definite compound might be isolated between $Sb_4O_5Cl_2$ and Sb_2O_3; and there is some evidence that such a compound had been analyzed.* With the hope of obtaining an intermediate product, we exposed to the direct sunlight during the summer months of 1876, a quantity of precipitated oxichloride of antimony, under a very large volume of water, which contained, besides the hydrochloric acid resulting from the decomposition of antimonious chloride, also a small quantity of tartaric acid. During this time there formed a considerable quantity of small acicular crystals, that were easily washed clean from the light amorphous precipitate with which they were mixed. The crystal appeared under the microscope perfectly homogeneous, and their surfaces had a brilliant vitreous lustre. They were rhombic prisms, having one of the two pairs of prismatic edges truncated by pinacoid planes. They presented, however, no distinctive terminations, sometimes tapering on the four prismatic planes to a point, and at other times on the pinacoids to an edge. They had the aspect of trimetric crystals, resembling some forms of Arragonite; and, as accurately as could be determined with the polarizing apparatus of a microscope, the principal optical sections were parallel to the prismatic edge, in whatever position the prisms might lie on the stage of the instrument. Nevertheless, these characters are not conclusive; and a delicate oblique striation which we observed on some of the prismatic planes led us to suspect that the crystals are really monoclinic. An analysis of the above crystals, made by Mr. Richardson, gave the following results: —

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* Gmelin's Hand-book of Chemistry, Cavendish edition, iv. 367.

† The decomposition of the antimonious oxichlorides by heat, after the manner we have previously described, affords the finest crystals of Sb_2O_3 we have ever seen. These crystals are in part brilliant octahedrons (Seuarmontite), but chiefly orthorhombic prisms (Valentinite). The last are frequently highly modified, and terminated at both ends; affording an opportunity for a more complete crystallographic investigation of this substance.

ANALYSIS OF ANTIMONIOUS OXICHLORIDE ($Sb_8O_{11}Cl_2$?).

No.	Weight taken.	Weight of AgCl obtained.	% of Chlorine.
1.	0.3605	0.0852	5.84
2.	0.7353	0.1759	5.91

Mean value 5.875

Theory for $Sb^8O^{11}Cl^2$ 5.88

It is therefore probable that we have here the intermediate compound sought; although it is important that these observations should be confirmed by further experiments. The assumed compound would be the second member of a possible series of oxichlorides, whose molecules each contain two atoms of chlorine, and of which the compound $Sb_2O_2Cl_2$ is the first term. In like manner, SbOCl is the first term of a parallel series, each of whose molecules contain one atom of chlorine, thus: —

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* Gmelin's Hand-book of Chemistry, Cavendish edition, iv. 367.

Oxichlorides of Antimony.

$SbOCl$	$Sb_2O_2Cl_2$
Sb_3O_4Cl	$Sb_4O_5Cl_2$
Sb_5O_7Cl	$Sb_6O_8Cl_2$
$Sb_7O_{10}Cl$	$Sb_8O_{11}Cl_2$

This table suggests, not only that there is a possibility of forming other compounds of this class, but also that there may be among them several isomers. On the next page, we give a table which offers a general review of the crystallographic relations of the more important antimonious compounds.

Our object in this paper has been to put on record a very considerable number of new facts; and if, in presenting them in their philosophical relations, we have laid open numerous deficiencies in our knowledge which must be supplied by future investigation, we have only made evident, in the case of antimony, what is equally true of our knowledge of the chemical relations of many other equally common elementary substances. Some of these deficiencies we hope to be able to supply ourselves in future papers.

We would again express our obligations to the same gentlemen named at the close of the previous paper, for the assistance they have rendered in this portion of the investigation also. We are especially indebted to Dr. Gooch, for his aid in the crystallographic measurements; and to Mr. Oliver W. Huntington, for the drawings with which the paper is illustrated.

Comparison of Crystalline Forms of Antimonious Compounds.

Orthorhombic.

	Valentinite	Sb_2O_3
	Stibnite	Sb_2S_3
	Stibiozincite	Sb_2Zn_3
	Dyscrasite	$SbAg_3$
	Antimonious Chloride	$SbCl_3$
	Antimonious Bromide	$SbBr_3$
α	Antimonious Iodide	SbI_3

Hexagonal.

β	Antimonious Iodide	SbI_3

Monoclinic.

	Antimonious Oxisulphide	Sb_2OS_2
	Antimonious Oxichloride	$Sb_4O_5Cl_2$
	Antimonious Oxibromide	$Sb_4O_5Br_2$
	Antimonious Oxi-iodide	$Sb_4O_5I_2$
	Antimonious Oxichloride	$SbOCl$
γ	Antimonious Iodide	SbI_3

III.

NOTE ON GRASSMANN'S CALCULUS OF EXTENSION.

By C. S. Peirce.

Read Oct. 10, 1877.

The last "Mathematische Annalen" contains a paper by H. Grassmann, on the application of his calculus of extension to Mechanics.

He adopts the quaternion addition of vectors. But he has two multiplications, internal and external, just as the principles of logic require.

The *internal* product of two vectors, v_1 and v_2, is simply what is written in quaternions as $-S.\,v_1\,v_2$. He writes it $[v_1 \mid v_2]$. So that

$$[v_1 \mid v_2] = [v_2 \mid v_1],$$

$$v^2 = (Tv)^2.$$

The *external* product of two vectors is the parallelogram they form, account being taken of its plane and the direction of running round it, which is equivalent to its *aspect*. We therefore have:—

$$[v_1 v_2] = v_1 v_2 \sin <^{v_1}_{v_2}.\ \mathrm{I}.$$

$$[v_1 v_2] = -[v_2 v_1], \qquad\qquad v^2 = o,$$

where I is a new unit. This reminds me strongly of what is written in quaternions as $-V(v_1 v_2)$. But it is not the same thing in fact, because $[v_1 v_2] v_3$ is a solid, and therefore a new kind of quantity. In truth, Grassman has got hold (though he did not say so) of an eightfold algebra, which may be written in my system as follows:—

Three Rectangular Vectors.

$$i = M : A - B : Z + C : Y + X : N$$

$$j = M : B - C : X + A : Z + Y : N$$

$$k = M : C - A : Y + B : X + Z : N$$

Three Rectangular Planes.

$$I = M : X + A : N$$

$$J = M : Y + B : N$$

$$K = M : Z + C : N$$

One Solid.

$$V = M : N$$

Unity.

$$1 = M : M + A : A + B : B + C : C$$
$$+ N : N + X : X + Y : Y + Z : Z$$

This unity might be omitted.

The relation of the two multiplications is exceedingly interesting. The system seems to me more suitable to three dimensional space, and also more natural than that of quaternions. The simplification of mechanical formulæ is striking, but not more than quaternions would effect, that I see.

By means of eight rotations through two-thirds of a circumference, around four symmetrically placed axes, together with unity, all distortions of a particle would be represented linearly. I have therefore thought of the nine-fold algebra thus resulting.

IV.

ON THE YOUNG STAGES OF SOME OSSEOUS FISHES.

By Alexander Agassiz.

Presented Oct. 10th, 1877.

I. *Development of the Tail.*

The structure of the tail of bony fishes has been described by Agassiz, Vogt, Owen, Stannius, Heckel, Huxley, Kölliker, and Lotz. The homocercal tails of bony fishes of the present day were contrasted by Agassiz and Vogt, and subsequently by Heckel, with those of the Ganoids, and of other fishes appearing before the Jurassic period, having so-called heterocercal tails; and they attempted to show that the tails of all homocercal fishes pass during their development, through a heterocercal stage. Heckel, Huxley, Kölliker, and Lotz have plainly shown, that, while the external appearance of the tail of bony fishes is homocercal, their real structure is only a modified heterocercal one: so that, as far as we now know, the tail of all fishes is built upon the modifications of the same type; the caudal fin not differing (as I shall show here), in its mode of development from the primitive embryonic fin, from that of the dorsal or anal fins. The theory of Agassiz, that the heterocercal tail of the young of bony fishes passes gradually into a homocercal one, and that the tail of the young of the bony fishes represents an embryonic stage which is permanent in Ganoids, is apparently overthrown by the well-established fact of the heterocercality of the tail of adult bony fishes modified externally so as to assume a homocercal form.

In the following notes, I shall describe the gradual change of the embryonic tail of several species of bony fishes, and call attention to the presence of an embryonic caudal lobe, which has thus far, apparently, escaped the notice of ichthyologists, and which shows remarkably well the identity of growth between the tails of Ganoids and of bony fishes.

As early as 1856, the late Professor Agassiz noticed in Lepidosteus a peculiar fleshy filament (the extension of the vertebral column) above the caudal fin, capable of independent vibratory motion; and compared the young caudal to a second anal, from its position as an appendage of the lower surface of the dorsal column.

This filament still exists in specimens having a length of eight inches. (See fig. 1.) Professor Wilder has lately * followed the transformations of the tail of the Gar-pike, from the time when the tail of the young Lepidosteus is in the stage corresponding to that of Pl. I. fig. 4, of this paper, until the filament has entirely disappeared, and the tail has assumed the rounded outline of the adult. He has also found traces of this filament in very young specimens of Amia, as a slight undulation of the dorsal edge of the caudal at the termination of the supposed notochord.

The stage figured by Wilder agrees very closely to the stage of Pl. I. fig. 10, of this note, and represents — I have not the least doubt, from having traced its gradual disappearance in so many of our bony fishes — the remnant of the fleshy embryonic filament of our bony fishes and of the Ganoids, as was surmised by Wilder.

I have given in Pl. I., quite in detail, the changes gradually taking place in the tail of a Flounder, from the time it leaves the egg until it has nearly assumed the final shape of the adult. On Plate II., I have given figures of a number of young fish tails, of different species, to show how general is the presence of the embryonic caudal fin, even in a comparatively advanced stage of growth. I have also given on the same Plate a figure of a young Lophius (Pl. II. fig. 9), a few days after its hatching from the egg, to show how extensive are the changes our fishes go through before reaching the adult condition; and I hope to give little by little, in papers I am now preparing, the general history of these changes in the principal families of our marine fishes, commencing with the development of the Pleuronectidæ.

In a young Pleuronectes, just hatched from the egg (Pl. I. fig. 1), the caudal end of the chorda is straight. It extends from the anterior arch between the otolites to its posterior extremity, in a line nearly parallel to the dorsal embryonic fin, nearer the dorsal than the ventral side. The embryonic caudal fin is rounded, and nearly symmetrical above and below, the dorsal fold being the narrowest.

In the next stage figured (Pl. I. fig. 2), the caudal extremity of the chorda has become slightly bent upwards, concave towards the ventral side; and then appears the first trace of the division line *f* between the embryonic and the permanent caudal fins; also, traces of the principal caudal rays, and of the accessory rays both of the dorsal and ventral side of the tail. In the following stage, the indentation between the embryonic and the permanent caudal has become deeper, the chorda more arched; the caudal fin-rays are well marked, and the permanent caudal now projects well beyond the general outline of the embryonic fin fold. The delicate lines imitating the fin-rays of the permanent fins, are specially prominent in subsequent stages (Pl. I. figs. 5–9). In the tail of the young of Pl. I. fig. 4, the whole tail is thrown up, and has now assumed the regular heterocercal type; the permanent fin-rays of the caudal being all placed on the lower side of the chorda. There is no trace, as yet, of any ossification of the vertebral column; the anterior and posterior supports of the fin-rays, as well as the supports of the accessory dorsal and ventral rays, are all cartilaginous.

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* Notes on the American Ganoids. Proc. Am. Ass. Adv. Sc., 1876, p. 153, Detroit Meeting.

In the next stages (Pl. I. figs. 5–7), the embryonic caudal has assumed the shape of a large independent lobe; the permanent fin proper extending entirely below it, and forming an independent fin, like a second anal, entirely on the lower surface of the notochord. The resemblance of the tail, at this stage, to the tail of Lepidosteus is so striking that I have here given, for comparison, a figure of the tail of the young Lepidosteus (eight inches in length), from which the late Professor Agassiz described the fleshy filament extending independently above the permanent caudal.

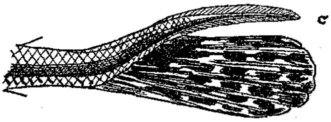

Fig. 1. Tail of young Lepidosteus.

In consequence of the greater arching of the notochord (Pl. I. fig. 5), and the simultaneous growth of the permanent caudal (the embryonic caudal remaining unchanged), the caudal fin is now bilobed. The principal permanent fin-rays of the tail-fin are well developed, but not yet articulated; the dorsal and ventral cartilages of the accessory fin-rays are well separated. In this and the preceding stages, large pigment spots are found between the two principal cartilaginous supports of the fin-rays, and afterwards greatly developed along their outer edge. The large black spot found on the tail of Amia, near the base of the caudal rays, recalls strikingly this space covered by pigment spots. We also find in other genera (Ctenolabrus), a large pigment spot, remaining more prominent than the others during the whole embryonic growth, and which can still be traced when the young fish has grown to a considerable size. In Ctenolabrus, the large pigment spot is placed on each side at the base of the tail, half-way between the termination of the dorsal and anal fins.

In Plate I. fig. 6, the caudal extremity of the chorda is still more arched upwards, the permanent caudal fin projects as far as the extremity of the embryonic caudal, the lower anterior edge is also separated by a slight indentation from the general line of the primitive embryonic fin-fold, and four or five of the principal caudal rays show a single articulation.

In this and in the subsequent stage (Pl. I. fig. 7), we see the first trace of the gradual disappearance of the embryonic caudal. In Pl. I. fig. 7, the permanent caudal projects beyond the embryonic tail, and there are traces of two articulations in a couple of the principal fin-rays. The permanent caudal has also gradually been thrown more upwards (Pl. I. fig. 8); the fin rays becoming more and more parallel with the axis of the body, until they gradually spread, fan-shaped, on each side (after passing through stage Pl. I. fig. 9) of a central line, as in Pl. I. figs. 10 and 11.

In Pl. I. fig. 8, we have the first sign of the disappearance of the extremity of the notochord, preparatory to the formation of the urostyle. (See Pl. I. figs. 10, 11, and 12.) The permanent caudal is now pointed, projecting far beyond the embryonic caudal, which, in the subsequent stage (Pl. I. fig. 9), is reduced to a slight lobe; it becomes still smaller in the next stage (Pl. I. fig. 10); and is finally reduced to a mere thickened semi-transparent edge, — the last remnant of the original embryonic fin-tail fold to be found in the permanent tail (Pl. I. fig. 11). Accompanying the disappearance of the embryonic caudal, we find a constant increase in the length of the permanent caudal (Pl. I. fig. 9). From being pointed, as in Pl. I. figs. 8, 9, it becomes somewhat rounded; and, with the more symmetrical arrangement of the fin-rays, the edge becomes scalloped as in figs. 10 and 11, and it does not differ materially from that of the adult in its general outline. In fig. 11, we can plainly see the ossification of the vertebræ, with the corresponding apophysis, the urostyle, the two principal cartilages supporting the fin-rays, with the dorsal and ventral cartilages supporting the accessory fin-rays. This is better shown in fig. 12, — a magnified drawing of the base of the tail of fig. 11.

The other genera of bony fishes in which I have traced the presence of this embryonic caudal lobe, or a trace of it, are Atherina, Batrachus, Cottus, Ctenolabrus, Lophius, Gasterosteus, Fundulus, Phycis, Gadus, Menhaden, Temnodon, Labrax, Scomber, six species of Pleuronectidæ, Poronotus, Lumpus, several of the genera of Viviparous fishes (Embiotocoidæ) from San Francisco, and a few other species as yet undetermined. In the very youngest specimens of Syngnathus I have been able to examine (fig. 2), the position of the two supports of the caudal rays, below the upturned termination of the notochord differed in no wise from that of the other bony fishes here mentioned.

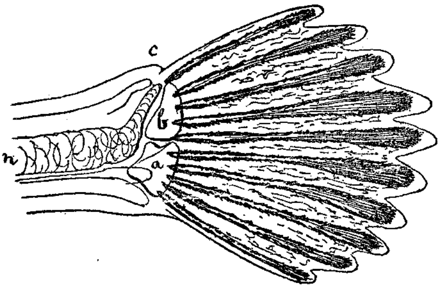

Fig. 2. Tail of young Syngnathus, ¾ in. long.

In Atherina, Pl. II. figs. 1–4, the embryonic caudal does not form quite so marked a lobe as in the species figured in Pl. I. Still, the separation between the permanent and the embryonic caudals is sufficiently well-marked to leave no doubt of the existence of the caudal lobe. The same is the case with Batrachus (Pl. II. fig. 5), with Lumpus (Pl. II. fig. 6), and Ctenolabrus (Pl. II. fig. 7). In the young Poronotus (Pl. II. fig. 8), the embryonic caudal lobe is more prominent.

In Lophius, the termination of the notochord remains unchanged quite late in life; the tail of the young Lophius (Pl. II. fig. 10) showing no trace of any ossification of the vertebral column, or degeneration of the extremity of the notochord, at a time when the young fish can readily be recognized as a young Lophius, from the presence of the peculiar appendages of the pectorals and of the anterior dorsal. In Gasterosteus (Pl. II. fig. 13), the embryonic caudal is again very prominent: it can readily be traced in Pl. II. figs. 14, 15, until the tail has assumed the shape it finally takes in the adult. In all the genera thus far described, the tail gradually passes from a strictly ventral appendage, placed below the dorsal column, to that of a terminal tail placed in the continuation of the vertebral column.

In Phycis and in the Cod, the structure of the tail is somewhat different; the accessory fin-rays, both of the dorsal and ventral side, are very numerous (Pl. II. figs. 19, 20), far outnumbering what are usually called the principal rays of the tail. These accessory rays early make their appearance (Pl. II. fig. 18); so that, although the terminal part of the chorda is turned up as in other fishes (Pl. II. figs. 19, 20), and the two principal cartilages of the tail-fin are placed below it, as in other fishes, yet, owing to this, the separation between the embryonic and permanent caudals is never distinctly indicated (Pl. II. figs. 18, 19, 20) — at least, not in specimens I have had the opportunity of examining — by an indentation or a sharp notch, as in other species figured in this paper. The tail of Phycis and of Gadus, therefore, which at first glance are so beautifully homocercal, do not in reality differ from the tails of other bony fishes; having like them a truly heterocercal termination (Pl. II. fig. 17), but completely disguised by the great development of the accessory fin-rays of the dorsal and ventral sides (Pl. II. figs. 19, 20).

In addition to the similarity of structure of the embryonic tail of bony fishes with that of the Ganoids, we find another point of comparison in the fleshy, fringed pectoral fins, which recall Huxley's Crossopterygians. This fleshy pectoral seems to be quite generally present in the embryos of bony fishes. It is represented for Lophius, on Pl. II. figs. 9, 11, 12. I have found similar fleshy, fringed pectorals in the embryos of Cottus and of several other bony fishes. In fact, immediately before the appearance of the rays in the pectorals, all bony fishes may be said to have such fringed, fleshy pectorals. They are, however, not sufficiently large and prominent to affect the general appearance of the young fish, except in the genera Lophius and Cottus, and one or two others; but in these genera they are well developed, and are similar in structure to the fleshy pectoral of young Lepidosteus.

Carus has called attention, in the Leptocephalidæ,* to the peculiar mode of termination of the chorda, — slightly bent upwards. The filament forming the tail-fin of Tilurus, the forked tail of some of the species of Leptocephalus, and the peculiar ending of their tail, recalling heterocercal tails, are certainly embryonic characters, — judging, at least, from similar stages of many of our common osseous fishes. These characters, with others, — such as the unossified chorda, the transparency of the body, the large prominent pigment spots, — all go far towards confirming the view of Carus, that the Leptocephalidæ are only the embryos of other fishes, such as Cepola and Trichiurus.

Both Huxley and Van Beneden † contend that the facts which they bring forward completely refute the theory of the parallelism of the embryonic tails of bony fishes with those of fishes preceding the Jurassic period: Huxley, because the bony fishes of the present day are not provided with a structurally homocercal tail (as was supposed by Agassiz and Vogt), but have — as he showed from Gasterosteus, and as I have shown here from many other genera of bony fishes — a truly heterocercal structure; Van Beneden, because in the Plagiostomes the tail of the young fish is at first truly homocercal, this condition preceding the heterocercal one, while, according to Agassiz and Vogt's theory, the young Plagiostomes should possess pre-eminently heterocercal tails; and, taking it also for granted that the oldest fossil fish known possessed truly homocercal tails, the whole theory, according to him, falls to the ground.

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* Carus, J. V., Ueber die Leptocephaliden. Leipzig, 1861, p. 13.

† Van Beneden sur le développement de la queue des Poissons Plagiostomes, Bull. Acad. de Belgique, 3me série, xl. No. 3.

Now, while fully admitting, with Huxley, that what Agassiz and Vogt called homocercal, in the modern bony fishes, is only an external delusion, due to a structure of the bones of the tail, which (as I have shown here) is found in a large number of bony fishes of the present day; while also admitting, with Van Beneden, that the young Plagiostomes, which in the adult have a truly heterocercal tail, yet have, in the early stages, a strictly homocercal (structurally also) tail, — yet I think that neither Huxley nor Van Beneden has upset the theory of Agassiz and Vogt; and that, mistaken as they were in the details, the great generalization remains, of the complete accordance between the embryonic growth and the paleontological development: only it must be carried one step farther; and we must, at the same time, give a somewhat different interpretation of the meaning of the heterocercality of the tail, so prevalent among the bony fishes of the present day, from that given to it by Agassiz and Vogt. Let us preface by stating that the heterocercal tail is not the earliest stage; and that neither Von Baer, nor Agassiz and Vogt, stated this, but merely noticed it as one of the early stages in the fish embryo. In fact, as is well known, the earliest stage of the tail in the egg, and immediately after hatching, is nearly symmetrical; the notochord extending in a straight line towards the tail, with the dorsal and ventral embryonic fins forming a rounded tail, the dorsal fin slightly narrower than the ventral. This stage we might call the Leptocardial (Pl. I. fig. 1), — the earliest form of tail assumed by bony as well as other fishes, which precedes that of the heterocercal tail proper (Pl. I. figs. 3, 4).

So that, as far as embryology is concerned, the tail of the Selachians is formed strictly in accordance with the law of development of other bony fishes; and it only remains to be seen how this accords with the paleontological record.

If we examine the tails of the Devonian fishes, — as we know them from the restorations of Agassiz, Hugh Miller, Pander, Heckel, Pictet, Huxley, and others, — we cannot fail to be struck with the exact parallelism of these ancient fishes, as far as the structure of the tail is concerned, with the structure of the successive stages of the tail of the young Flounder, figured on Plate I. of this paper.

We find, among the Devonian fishes, genera with truly leptocardial tails, like those of Pl. I. fig. 1, such as the genera Glyptolæmus, Gyroptichius; also, genera with slightly modified tails (with the least possible tendency to heterocercality), as in figs. 2 and 3, — Holoptichius and Osteolepis; next, such genera as Glyptolepis, where the heterocercal tail is somewhat more marked, approaching nearer the form of Pl. I. fig. 4. But it must be remembered, that in all these genera, although the outline of the tail-fin is much as in the figures here given (Pl. I. figs. 2–4), yet, as in Polypterus and still more in Ceratodus, the scales extended over the dorsal column into the tail, in a triangular shape; and it is only in such genera as Dipterus that the heterocercal character becomes more prominent, as in Pl. I. fig. 4.

This does not by any means conclude the parallelism, which is still more striking when we come to such forms as Phaneropleuron and Tristichopterus, where the tail is lobed, the dorsal column extending into the dorsal lobe exactly as in the stage represented in Pl. I. figs. 5, 6.

In the Old Red, such genera as Acanthodes, Diplacanthus, Cheirolepis, and the like, represent stages corresponding to those of Pl. I. figs. 5, 6, 7, where we find the first indication of the separation of a true caudal and of an embryonic caudal. In the subsequent modifications of the tail of fossil fishes (approaching Lepidosteus), the tendency has been gradually to lessen the upper embryonic caudal lobe, and to give greater prominence to what is to become the caudal proper; although there is not, of course, the difference in structure of the fin-rays of the two sections to separate them, as in bony fishes. It is only when we compare these older forms with such genera as Platysomus, Semionotus, Lepidotus, and finally Pachycormus, that we trace the gradual approach to an externally homocercal tail, much by the same process which we readily follow in the embryo fish through the corresponding changes from Pl. I. fig. 8, to Pl. I. fig. 11. The gradual shortening of the extremity of the chorda dorsalis, until it only extends slightly in advance of the base of the caudal rays, is strictly analogous to the disappearance of the embryonic caudal and the gradual development of the permanent caudal from the lobes of the heterocercal tail of the young fish embryo.

In addition to the parallelism of the embryonic and paleontological development of the tail, we find other embryonic characters in the Old Red fishes. I have already alluded to the old-fashioned structure of the pectorals in the embryo of Lophius and other bony fishes; and would call attention to the innumerable embryonic fin-rays (Pl. I. figs. 5–9) of the embryonic dorsals and ventrals, which recall strikingly the similar numerous rays so characteristic of the fins of the older Ganoids.

So that, while Agassiz and Vogt were undoubtedly mistaken in the details of their explanation and comparison of the homocercal and heterocercal tails, yet the parallelism they attempted to prove, not only exists, but can even be carried out far beyond any thing they conjectured. Their very mistakes regarding the heterocercal structure of what they called a homocercal tail in the bony fishes of the present day being (as I have attempted to show) the best proof of the existence of such a parallelism, and the clearest indication possible of the uniformity of structure of the tails of the fishes of the present day with those of the fishes of the most ancient geological period in which fishes have as yet been found. This parallelism could, however, not be conclusively made out, until it was proved that the extension of the chorda dorsalis into the upper lobe of the heterocercal tail gave us the explanation of the peculiar heterocercal structure still to be traced in the so-called homocercal tails of the bony fishes of the present day, long after the disappearance of the upper caudal lobe, which (as I have shown) exists in bony fishes only during a short embryonic period.

DESCRIPTION OF THE PLATES.

LETTERING.

a.	Anterior cartilage on lower side of notochord, supporting principal fin-rays.
b.	Posterior cartilage on lower side of notochord, supporting principal fin-rays.
a′.	Anterior dorsal cartilage.
b′.	Posterior dorsal cartilage, supporting accessory fin-rays.
c.	Embryonic caudal fin.
d d.	Principal caudal rays.
d′.	Accessory dorsal caudal rays, above notochord.
d″.	Accessory ventral caudal rays, below notochord.
df.	Dorsal fin.
vf″.	Ventral fin.
f.	Permanent caudal fin.
n.	Notochord.
v.	Last ossified vertebra.
v′.	Dorsal apophysis.
v″.	Ventral apophysis.
w.	Urostyle.

PLATE I.

TAIL OF FLOUNDER.

Fig. 1. Tail of young fish, with straight notochord and embryonic fin.

„ 2. Slightly older — the extremity of notochord somewhat arched, and showing first trace of caudal fin.

„ 3. The indentation between the embryonic caudal and the permanent caudal is deeper; fin-rays well defined.

„ 4. Extremity of notochord still more arched than in preceding figures; the separation between the permanent and embryonic caudals somewhat more distinct.

„ 5. In this stage, the permanent and embryonic caudals form a sharp angle; the distinction between embryonic and permanent rays is well shown.

„ 6. The permanent caudal extends as far as the embryonic caudal, which now shows traces of resorption.

„ 7. The pointed permanent caudal extends beyond the line of the embryonic caudal, somewhat decreased in size.

„ 8. The cartilaginous supports of the fin-rays proper have become large; the extremity of the notochord shows traces of the formation of the urostyle.

„ 9. The permanent caudal has increased greatly in length; the embryonic caudal is now reduced to a small rounded lobe.

„ 10. The caudal has become well rounded; a mere trace of the embryonic caudal is left. The urostyle is also more distinct.

Fig. 11. The tail has now the shape of that of the adult; the merest trace of the embryonic caudal remains.

„ 12. Fig. 11 somewhat magnified, to show the cartilages supporting the permanent and the accessory fin-rays of the tail, as well as the urostyle with last ossified vetebra.

The young Flounder of Fig. 1 measured $6^{\text{mm.}}$ in length; that of Fig. 12 measured $18^{\text{mm.}}$; the whole change of the tail, from the straight notochord of Fig. 1 to the rounded tail of Fig. 12, took place in about three weeks, judging from the specimens fished up.

PLATE II.

Tails of Embryos of Young Fishes.

Figs. 1–4. Atherina, respectively $5^{\text{mm.}}$, $9^{\text{mm.}}$, $10^{\text{mm.}}$, $11^{\text{mm.}}$ long.
Fig. 5. Batrachus. $9^{\text{mm.}}$ long.
„ 6. Lumpus. $4^{\text{mm.}}$ long
„ 7. Ctenolabrus. $6^{\text{mm.}}$ long.
„ 8. Poronotus. $7^{\text{mm.}}$ long.
„ 9. Young Lophius with straight notochord. About $5^{\text{mm.}}$ long.
„ 10. Heterocercal tail of young Lophius. $20^{\text{m.}}$ long.
„ 11. Fleshy pectoral of same, seen from the side.
„ 12. Same seen from above.
Figs. 13, 14. Gasterosteus.
„ 16, 17, 18. Phycis. Fig. 16 *f.* $3^{\text{mm.}}$, $8^{\text{mm.}}$, $15^{\text{mm.}}$.
Fig. 19. Cod. $20^{\text{mm.}}$ long.
„ 20 and Fig. 18. Magnified. (Phycis.)

V.

PRELIMINARY WORK ON THE DETERMINATION OF THE LAW OF THE PROPAGATION OF HEAT IN THE INTERIOR OF SOLID BODIES.

By B. O. Peirce, Jr., and Edward B. Lefavour.

Presented Oct. 10, 1877.

For a long time it has seemed probable, as stated in a paper[*] published last spring in the Proceedings of the Academy, that the flux of heat in any direction x, in a solid body, can be written, —

$$-c\frac{df(v)}{dx}$$

where v is the temperature, c a constant different for different substances, and $f(v)$ an undetermined function of v. The object of this paper is to show that such a function $f(v)$ can be found.

This is not of necessity possible; for denoting by X, Y, Z, the components in three rectangular directions of the vector function which represents the flux, we have

$$-c\frac{dF(x, y, z)}{dx} = X$$

$$-c\frac{dF(x, y, z)}{dy} = Y$$

$$-c\frac{dF(x, y, z)}{dz} = Z$$

whence

$$-c\, dF(x, y, z) = \mathrm{X}\, dx + Y\, dy + Z\, dz$$

This equation is integrable, and $F(x, y, z)$ can be found only when

$$\frac{dY}{dx} = \frac{dX}{dy} \qquad \frac{dZ}{dy} = \frac{dY}{dz} \qquad \frac{dX}{dz} = \frac{dZ}{dx}$$

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[*] Note on the Determination of the Law of Propagation of Heat in Solid Bodies.

Our experiments were directed to determining whether these conditions can be satisfied, and

$$F(x, y, z) = f(v).$$

When a body heated in any way reaches a final state, — that is, a state where just the same quantity of heat enters each portion during a given time as leaves it, — the function $f(v)$, if it exists, must satisfy the equation

$$\frac{d_x^2 f(v)}{dx^2} + \frac{d_y^2 f(v)}{dy^2} + \frac{d_z^2 f(v)}{dz^2} = 0.$$

One, and only one, solution of this equation corresponds to each set of physical conditions. Since v, and any function of v, are constant along the same surfaces, if, when the body is in a final state, v is constant along each surface of the family $\varphi\,(x, y, z) = k$, where $\varphi\,(x, y, z)$ is the solution of Laplace's equation, corresponding to the given physical conditions, then it is always possible to find a function of the temperature alone, which shall satisfy Laplace's equation as a function of x, y, z, or, what is the same thing, shall be equal to $\varphi\,(x, y, z)$ throughout all space.

For let u and v be two functions of x, y, z, such that $u = k$, and $v = c$ represent the same family of surfaces, then denoting by du the total differential of u, and by $d_x u$ the partial differential relative to x,

$$\frac{\frac{d_x u}{dx}}{\frac{d_y u}{dy}} = \frac{\frac{d_x v}{dx}}{\frac{d_y v}{dy}}, \qquad \frac{\frac{d_y u}{dy}}{\frac{d_z u}{dz}} = \frac{\frac{d_y v}{dy}}{\frac{d_z v}{dz}} \qquad \frac{\frac{d_z u}{dz}}{\frac{d_x u}{dx}} = \frac{\frac{d_z v}{dz}}{\frac{d_x v}{dx}},$$

$$\therefore \quad \frac{\frac{d_x u}{dx}}{\frac{d_x v}{dx}} = \frac{\frac{d_y u}{dy}}{\frac{d_y v}{dy}} = \frac{\frac{d_z u}{dz}}{\frac{d_z v}{dz}} = \psi\,(x, y, z).$$

If p is any variable,

$$\frac{d_p u}{dp} = \frac{d_x u}{dx} \cdot \frac{d_p x}{dp} + \frac{d_y u}{dy} \cdot \frac{d_p y}{dp} + \frac{d_z u}{dz} \cdot \frac{d_p z}{dp}$$

And

$$\frac{d_p v}{dp} = \frac{d_x v}{dx} \cdot \frac{d_p x}{dp} + \frac{d_y v}{dy} \cdot \frac{d_p y}{dp} + \frac{d_z v}{dz} \cdot \frac{d_p z}{dp}$$

or, substituting,

$$\frac{\frac{d_p u}{dp}}{\frac{d_p v}{dv}} = \psi\,(x, y, z).$$

Similarly the ratio of the corresponding total differential coefficients is

$$\frac{\frac{du}{dp}}{\frac{dv}{dp}} = \psi\,(x, y, z).$$

Whence, changing the variable and integrating,

$$\frac{d_v u}{dv} = \psi\,(x, y, z) = \frac{du}{dv}$$

The partial and total differential coefficients of u taken relatively to v cannot be equal, if u involve any other quantity than v,

$$\therefore \quad u = f(v)$$

In short, when a body is heated in any manner whatever, there must exist a function $f(v)$, the same for all bodies, whose derivative in any direction, when multiplied by a constant depending on the nature of the body, gives the flux of heat in that direction, provided v is found constant along the surfaces $\varphi\,(x, y, z) = k$, which belong to the solution of Laplace's equation for that particular case.

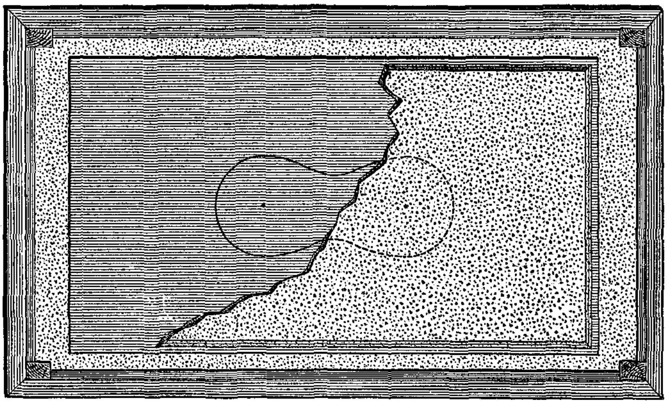
The first case open to direct and satisfactory experiment is where a plate of metal is heated at two points, and exposed to the air only at its edges. The isothermals in this case belong to the family $A \log r_1 + B \log r_2 = k$;

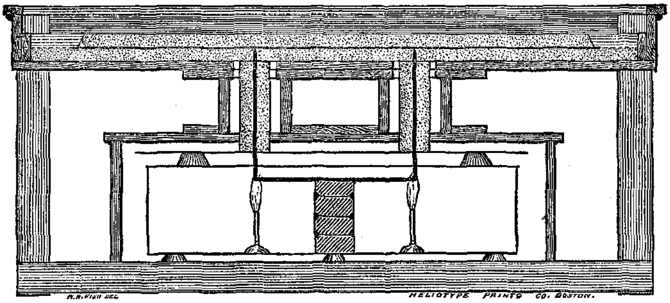
or $$r_1 r_2^{\,m} = c.$$

This latter form of the equation shows that the two points need not be *heated equally*. The solution for three dimensions cannot be readily submitted to experiment; but the probability that $f(v)$ should satisfy the solution for two dimensions, and fail in that for three, is so slight that it may be neglected.

Our first experiments were with a small iron plate covered with a mixture of wax, rosin, and paraffine. By this means, one curve was obtained at each heating; viz., that separating the part of the mixture which had melted from that which remained solid. This method can be used in the open air, but is impracticable when the waxed plate is covered by a non-conducting material.

After some further rude work, we constructed the table of which a diagram is given below:—





H.R.FISH DEL. HELIOTYPE PRINTG CO. BOSTON.

Two Bunsen burners are enclosed in an iron case, which is itself enclosed in a wooden case, surrounding it at a distance of 5 to 7 cm. Through the top of the iron case is put a bent copper rod, heated at either elbow by one of the Bunsen flames. The wooden case rests on a floor fastened beneath the table. The ends of the rod, rising from the iron case, and surrounded and held firmly in place by tin cylinders closed at both ends, and themselves secured to enclosing wooden pipes, project some 5 cm. above the top of the table.

Around this top is a guard, within which are wooden cleats, by means of which to level the non-conducting material. This non-conductor is the almost pure silicious earth, dug in Keene, N. H., and known as "infusorial earth." With it, we were abundantly supplied by the kindness of Messrs. J. A. Wright and James H. Wilson of Keene. It is in every way suited to work of this kind, being clean, free from any appreciable amount of moisture, and an almost perfect non-conductor of heat.

Upon the ends of the copper rod, covered above and below to within 2 or 3 cm. of the edges, rests a sheet of No. 11 boiler-plate iron, 1.5 meters long and .9 meters wide.

The first thing necessary to the success of our experiments with the plate was that the head of gas should be constant. We found the variation to be insensible, although a change in pressure of a fraction of one mm. of water could have been detected.

The temperature of the air must also be constant. To secure this, we were obliged to use the precautions mentioned above. The air for the lamps entered at the bottom of the cases, and passed out from both by an iron pipe placed within a wooden one. The tin cylinders were filled with "infusorial." The variation in the temperature of the air was less than one degree during the day. It would have been impossible to keep the temperature exact to the $\frac{1}{1000}$ of a degree, as is reported of Biot.

No work could be done until the plate reached a final state. For this an average of five hours was required. After this time the temperatures were sensibly constant.

The large size of the plate was necessary, because observations taken near the edge ought not to be relied on, and because the variation in temperature was quite rapid, whilst even a slight error in determining a point on the curve produced a great change in its equation. For example, in one case, in a curve whose loop was about 210 mm. in diameter, a change of *four* millimeters in the position of a given point changed m in the equation $r_1 r_2^m = c$ from $-.5$ to $+.8$, or the equation from

$$\frac{r_1}{r_2^{\frac{1}{2}}} = c \text{ to } r_1 r_2^{\frac{4}{5}} = c.$$

In another case, a change of three millimeters altered the equation from

$$r_1 r_2^3 = c \text{ to } r_1 r_2^{\frac{1}{2}} = c$$

The measurements were made with the galvanometer and thermopile. Our galvanometer has four coils, whose resistance is 11. ohms, and a mirror of 1 meter focal length. Each scale-division is $\frac{5}{8}$ mm.

In order that the electromotive force of the current, and therefore the deflections of the galvanometer, should be directly proportional to the excess of temperature of the point touched over that of the air, a pair of metals must be taken whose neutral point is at a very high or very low temperature, and whose relation to each other varies the least possible with the temperature. Iron and German-silver satisfy these conditions remarkably well. Their neutral point is at 1354°, as deduced from Prof. Tait's table. Some rude experiments of ours on the relation between the temperature and the deflections gave quite ragged curves; but they were evidently in every case straight lines. The three lines obtained by heating and cooling water were nearly parallel. The mean of these experiments showed that one degree of temperature corresponds to 4.3 scale-divisions. One microvolt corresponds to 2.6 scale-divisions, or one scale-division to .39 microvolts. We thus find that one degree of change in temperature produces an electromotive force of 18.3 microvolts. The value as calculated by Prof. Tait's formula is 21.8.

Our first experiments were with two thermopiles. The method of work was this. Each thermopile consisted of one joint, which was fitted to a piston moving vertically 3 cm. in a brass cylinder. The piston was depressed by a weight resting on a lever, whose fulcrum was on the frame around the table, and its end at the thermopile. The average pressure applied was 9 kilos. This was opposed by the reaction of springs, which raised the piston when the weight was removed. The cylinders were placed on carriages, running on wooden strips, which crossed the table transversely just above the infusorial. These strips and the sides of the table were graduated, and thus we got the (x, y) co-ordinates of the points touched.

One thermopile was kept in a fixed position, the other was moved about to various points where equal deflections were obtained. All these last points lie on an isothermal. Great care is necessary that the contact with the plate be good, and that the piston move in an exactly vertical direction. The results thus found were unsatisfactory, because, as before pointed out, an inappreciable error in observation produces a very perceptible change in the equation of the curve. Another objection is that the method requires a great expenditure of time.

A second method is more satisfactory. One thermopile was moved lengthwise along the plate, and the deflections laid off in a curve. The same may also be done transversely, but the curves obtained are of little value.

The results of our work on "longitudinal curves" are given below. Evidently, in crossing the plate, we shall find, in general, four points having the same temperature, two about either pole. The isothermal points lying about one pole in each of two experiments, obtained by averaging several readings of the curves mapped out, are as follows: —

Isothermal Points.

Exp. 1.	Exp. 2.	Mean.
488 — 658.1	488 — 661.9	488 — 660.
495 — 644.4	495 — 643.4	495 — 643.9
500 — 632.6	500 — 631.1	500 — 631.85
502½ — 626.8	502½ — 625.9	502½ — 626.35
505 — 622.5	505 — 620.6	505 — 621.55
507½ — 617.4	507½ — 615.9	507½ — 616.65
510 — 612.9	510 — 611.8	510 — 612.35
515 — 604.4	515 — 602.3	515 — 603.85
517 — 600.9	517 — 599.4	517 — 600.15
520 — 596.9	520 — 595.6	520 — 596.25
525 — 591.3	525 — 588.8	525 — 590.05
527 — 588.4	527 — 586.6	527 — 587.5
530 — 584.6	530 — 582.5	530 — 583.55
535 — 578.7	535 — 576.2	535 — 577.45
555 — 555	555 — 555	555 — 555

The isothermal curves corresponding to these pairs of points are in general form like the family to which the common lemniscate belongs. Several other curves have much the same form, and therefore it becomes necessary to show that our experimental curve is none of these. The following conditions must be satisfied: —

(1) For a given positive value of r_1 there must be only one real and positive value of r_2, r_1 and r_2 being radii from the fixed points.

(2) The function must be homogeneous; else, for the same physical problem we shall get as many different solutions as we employ units of measure.

Hence we must exclude trigonometric functions, by reason of their periodicity, and, since exponential and logarithmic forms may be developed in algebraic series, need consider only these last.

In this system of coördinates, negative, as well as truly imaginary values of r_1 and r_2, are to be regarded as imaginary. (1) requires us to exclude any form in which both positive and negative terms enter, as also forms where the exponents of the powers of r are both positive and negative.

Three simple equations satisfy these conditions:—

$$(1.)\ r_1 r_2 = c. \qquad (2.)\ \frac{1}{r_1} + \frac{1}{r_2} = c. \qquad (3.)\ r_1 + r_2 = c.$$

(3) represents the ellipse; and all forms $r_1^m + r_2^m = c$ are of equally little use to us.

The isothermal points for $\frac{1}{r_1} + \frac{1}{r_2} = c$ are approximately:—

488 — 627
495 — 620
500 — 615½
502½ — 612
505 — 608½
507½ — 605
510 — 602½

515 — 595½
517 — 593
520 — 590
525 — 585½
527 — 583½
530 — 580
535 — 575

The isothermal curves for $\frac{1}{r_1^2} + \frac{1}{r_2^2} = c$ are much nearer circles, and the isothermal points more nearly equidistant from the pole. This approximation to a series of circles increases with the degree of the equation. There remains to be considered only $\Sigma\, r_1^{m+x} r_2^{m-x} = c$.

The physical conditions require that the two loops of the isothermal curves shall be approximately equal. But, whenever the ratio $\frac{m+x}{m-x}$ in the equation $r_1^{m+x} r_2^{m-x} = c$ differs much from 1, the loops are very unequal; and any term in which a ratio of this sort is found introduces into the equation $\Sigma\ r_1^{m+x} r_2^{m-x} = c$ an inequality unbalanced by the other terms.

Assuming now $r_1 r_2^p = c$, as the equation of our experimental curve, the values of p, corresponding to the isothermal points in the "Mean" column, are as follows:—

(1)	1.097	(8)	.988	average 1.059
(2)	1.125	(9)	.912	
(3)	1.093	(10)	.950	
(4)	1.072	(11)	1.056	
(5)	1.065	(12)	1.089	
(6)	1.041	(13)	1.098	
(7)	1.035	(14)	1.207	

Compare now the experimental values before given with the four series of values below.

The third and fourth fail to satisfy condition (2), as this condition was discovered too late to change our figures, but they will serve equally well for illustration.

Values corresponding to	in $r_1r_2^{1.05}=c$*	in $r_1r_2^{1.075}=c$*	in $r_1r_2^{.9}+r_1r_2^{1.2}=c$†	in $r_1r_2^{.9}+r_1r_2^{1.25}=c$†
488	657.1	658.5	662.	668.
495	640.8	641.8	643.	649.
500	630.6	631.3	632.	637.
502½	625.8	626.4	625.	630.5
505	621.2	621.8	620.5	625.
507½	616.8	617.3	616.5	620.
510	612.6	613.	612.	615.
515	604.6	604.8	602.	606.5
517	601.5	601.8	600.	602.5
520	597.1	597.3	595.	598.
525	590.	590.2	588.	591.5
527	587.3	587.4	585.5	588.
530	583.4	583.5	581.	583.
535	577.1	577.15	575.5	577.

It will appear at once that while the first two curves lie very close to the experimental curve, and parallel with it, even such close approximations to the one-term equation as those above cut at an angle. The use of residual curves will make the fact still clearer. Equations of three, four, or more terms, will yield like results.

Even if an equation could be found of this kind which was equally good with the simple form, it would still be improbable that the simpler should not be the form preferred by nature.

Since then the temperature is shown experimentally to be constant along the family of curves $r_1^m r_2 = c$, a function $f(v)$ can be found, the same for all bodies, whose derivative in any direction in a given body, when multiplied by a constant quantity depending on the nature of the body, measures the flux of heat in that direction.

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* Computed. † Approximate.

The solution for one dimension $\left[\frac{d_x^2 f(v)}{dx^2} = 0\right]$, as well as that for two, admits of experimental verification.

Let a long rod, covered with non-conducting material, except at the ends, be heated at the points A and B. Then for A: $f(v_1) = k_1 r_1 + b_1$. For B: $f(v_2) = k_2 r_2 + b_2$; r_1 and r_2 being the distances from A and B measured on the rod.

$$f(v) = f(v_1) + f(v_2) = k_1 r_1 + k_2 r_2 + b_1 + b_2 = k_1 r_1 + k_2 r_2 + \lambda.$$

$$\text{Let } k_1 = k_2 \qquad fv = k_1(r_1 + r_2) + \lambda.$$

$f(v)$ between A and B is constant, and may be represented by a straight horizontal line. Whenever $f(v)$ has a constant value, v has a constant value, and conversely. Therefore, v, as well as $f(v)$, must be represented between A and B by a straight line. If A and B are not heated exactly alike, the line will still be straight, but inclined to the horizontal.

We have not yet tried the experiment, but intend to do so, that there may be the greatest possible assurance in regard to the existence of $f(v)$, seeing that it exists in every case which can be experimentally tested.

The second portion of our work is the determination of $f(v)$. For a rod heated to the final state, it must satisfy the condition $\frac{d_x^2 f(v)}{dx^2} = 0$, or $f(v) = Ax + B$.

For a plate: $\frac{d_x^2 f(v)}{dx^2} + \frac{d_y^2 f(v)}{dy^2} = 0$, or $f(v) = C + D \log r$ (changing to polar co-ordinates).

For a solid: $\frac{d_x^2 f(v)}{dx^2} + \frac{d_y^2 f(v)}{dy^2} + \frac{d_z^2 f(v)}{dz^2} = 0$, or $f(v) = E + \frac{F}{r}$.

This function cannot be v itself. For, in the case of a rod, heated at one point, let $fv = v$.

$$\begin{array}{ll} & v = Ar + B. \\ \text{When } r = 1 & v = A + B \neq \infty \\ ,, \quad r = 0 & v = B \neq \infty \end{array} \left\{ \begin{array}{l} \text{since by physical conditions} \\ \text{the temperature must here} \\ \text{be finite.} \end{array}\right.$$

$$,, \quad r = \infty \quad v = A\,\infty + B = 0, \text{ which is impossible.}$$

For a plate

$$\begin{array}{ll} f(v) = v = C + D \log r & \\ \text{When } r = 1 & v = A \neq \infty \\ ,, \quad r = 0 & v = C + (-\infty), \end{array}$$

which again is contrary to the physical conditions.

For a solid

$$f(v) = v = E + \frac{F}{r}$$

When $r = \infty \quad v = 0 \quad \therefore E = 0 \quad \therefore v = \frac{F}{r}$

„ $r = 0 \quad v = \frac{F}{r} = \infty$, which again is impossible.

The same thing may be proved as follows. Assume, for the sake of argument, that the flux is $-c\frac{d_x v}{dx}$ for the direction x, $-c\frac{d_y v}{dy}$ for y, $-c\frac{d_z v}{dz}$ for z. Then for all homogeneous bodies,

$$\frac{dv}{dt} = \lambda\left(\frac{d_x{}^2 v}{dx^2} + \frac{d_y{}^2 v}{dy^2} + \frac{d_z{}^2 v}{dz^2}\right).$$

This is the only condition that a function v must satisfy, in order to represent an actual possible case. Any function that is a particular solution of this equation may represent an actual distribution of heat.

First. If $v = \varphi(x, y, z, t)$ represent the temperature throughout a body, it cannot have any true maxima and minima for x, y, z. This is evident, since the conditions of a maximum or minimum are $\frac{d_x v}{dx} = 0$, $\frac{d_y v}{dy} = 0$, $\frac{d_z v}{dz} = 0$, which by hypothesis cannot be zero without making the flux $= 0$. No point can be hotter or colder than the points around it, and there not be a flux to or from the point. The physical conditions forbid the mathematical condition of the existence of maxima and minima. There must be points hotter than points around them, and therefore they must be shooting points, and not maxima or minima.

Secondly. If v_1 and v_2 are two particular solutions of the Partial Differential Equation, their sum is also a solution, and therefore corresponds to an actual distribution of heat, in which the temperature of any point is equal to the sum of the temperatures, which it would have under the conditions represented by v_1 and v_2. It will now be easy to show

Thirdly. The points of hottest temperature, when the solution is v, fall in exactly the same places in the body as the hottest points of the two solutions v_1 and v_2 taken jointly. That is to say: if there are n_1 points of highest temperature when the solution is v_1, and n_2 when the solution is v_2, there will be $n_1 + n_2$ points of highest temperature when the solution is $v_1 + v_2 = v$, unless v_1 and v_2 have some hot points in common, or else a minimum of v_1 or v_2 corresponds with a maximum of v_2 or v_1. Here special investigation is necessary for each particular case.

Observe carefully that, if the functions v, v_1 and v_2 attained *true* maxima or minima at the hot points, instead of coming to a point, the proposition would not be true.

That the proposition is true, in regard to shooting points, can be seen thus: $\frac{d_x v}{dx} = \tan \tau$; $\tan \tau$ changes sign instantaneously at a shooting point, and hence, for a small (infinitely small) change in $x[dx]$, $\frac{d_x v}{dx}$ has a finite change of value, and $d_x \frac{d_x v}{dx} =$ finite, $dx = \frac{1}{\infty}$. Therefore

$$\frac{d_x \frac{d_x v}{dx}}{dx} = \frac{\textit{finite}}{\frac{1}{\infty}} = \infty$$

This is the condition for a shooting point. If, at the same point, $\frac{d_x v}{dx}$ as well as $\frac{d_x^2 v}{dx^2}$ is infinite, there is a cusp. If also $v = \infty$, there is an asymptote.

At the points of highest temperature $\frac{d_x^2 v}{dx^2} = \infty$, $\frac{d_y^2 v}{dy^2} = \infty$, $\frac{d_z^2 v}{dz^2} = \infty$.

$$\text{Now } \frac{d_x^2 v}{dx^2} = \frac{d_x^2 v_1}{dx^2} + \frac{d_x^2 v_2}{dx^2}$$

and any values of x, y, z, which make either term in the second member infinite, will make the whole member infinite, and insure a shooting point for v as far as the direction x is concerned, wherever there is one for either v_1 or v_2. The like is true of y and z. If $\frac{d_x^2 v_1}{dx^2}$ and $\frac{d_x^2 v_2}{dx^2}$ become ∞ with opposite signs, there will be an ambiguity which can be got rid of by the determination of the indeterminate quantity $(\infty - \infty)$.

Using the results of both (2) and (3), it is easy to arrive at an absurdity in almost any case considered. For instance, heat to a constant temperature all the points upon a circle, marked out upon a large metal plate protected from surface radiation. Then not only will the point at the centre of the circle soon become hotter than any point upon the rim, in which case the heat must flow from within out, but it must finally be infinitely hot, which is absurd.

Our experiments have not proceeded so far that we can determine

$f(v)$ from them. That determination must form the subject of a subsequent paper. We will simply present here a specimen of our experimental results for both the rod and the plate, heated at one point under the conditions above specified; the values in column "v" being galvanometer deflections: —

Rod.		Plate.	
r	v	r	v
0	342	0	178
100	257	50	108
200	196	100	83
300	150	150	$61\frac{1}{2}$
400	111	200	53
500	87	300	37
600	67	400	$28\frac{1}{2}$
700	54	500	$20\frac{1}{2}$
800	43	600	15
900	33		
1000	29		

From these series we are to determine v as a function of r. If $v = \varphi(r)$. Then for a rod, $f\varphi(r) = Ar + B$; for a plate, $f\varphi(r) = C + D \log r$.

Permit us here to express our great gratitude to your Academy for the generosity and liberality which have supplied us with the means for carrying on our experiments.

HARVARD UNIVERSITY, Oct. 10, 1877.

VI.

PROBABILITIES AT THE THREE-BALL GAME OF BILLIARDS.

By Benjamin Peirce.

Read Oct. 10, 1877.

In the three-ball game of billiards, the person who makes a successful shot adds one to his counts. In case of a discount, the person who gives the discount loses, moreover, one from his count, when his opponent makes a successful shot. In the case when he gives a double discount, he loses two for each successful shot; and, in the same way, for a treble, quadruple, &c., discount, he loses three, four, &c., points from his count. In the grand discount, he loses all which he may have made, whenever his opponent succeeds in his shot. Whenever a player fails in his shot, the other player takes the cue.

Let the two players be A and B, and let h be the whole number of points of the game. Let a be the probability that A will make his shot, and b the probability that B will make his shot. No allowance is made for the increase of probability of a successful shot after the first shot, although this is a very important consideration with good players. It may justly be thought that the failure to recognize this change of probability reduces the practical value of the investigation. But imperfection is inevitable in the earlier stages of any research.

Let, then,
$$A = \frac{a}{a+b-ab}, \quad B = \frac{b}{a+b-ab},$$
so that
$$Ab = Ba = A + B - 1,$$
$$(1-a)\,(1-b) = 1 - \frac{a}{A} = 1 - \frac{b}{B}.$$

Let A give n discounts to B. When A needs i more points to make the game, and B needs j more points, let

$F(i, j)$ be A's probability of winning when he has the play, and

$f(i, j)$ be A's probability of winning when B has the play.

The fundamental equations are obviously

$$F(i,j) = aF(i-1,j) + (1-a)\,f(i,j),$$

$$f(i,j) = bf(i+n,j-1) + (1-b)\;F(i,j),$$

in which $i+n$ must be reduced to h whenever it exceeds h. Substitution, transposition, and division give at once

$$\begin{aligned} F(i,j) &= AF(i-1,j) + (1-A)\,f(i+n,j-1) \\ &= AF(i-1,j) + BF(i+n,j-1) - aBF(i+n-1,j-1). \end{aligned}$$

When no discount is given, this equation becomes

$$F(i,j) = AF(i-1,j) + BF(i,j-1) - aBF(i-1,j-1),$$

which is an especial case of an equation solved by Laplace in his Calculus of Probabilities.

When it is a grand discount, or whenever

$$i \succ h - n,$$

the equation becomes

$$\begin{aligned} F(i,j) &= AF(i-1,j) + (1-A)\,f(h,j-1) \\ &= AF(i-1,j) + BF(h,j-1) - aBF(h-1,j-1). \end{aligned}$$

These are special cases, whatever may be the discount: —

$$\begin{aligned} F(i,\ 0) &= f(i,\ 0) = 0, \\ F(0,\ j) &= f(0,j) = 1, \\ F(1,\ 1) &= A, \\ F(i,\ 1) &= AF(i-1,\ 1) = A^i. \end{aligned}$$

In the case of the grand discount,

$$\begin{aligned} F(i,j) - f(h,j-1) &= A\,[F(i-1,j) - f(h,j-1)], \\ &= A^i[1 - f(h,j-1)], \\ F(i,j) &= A^i + (1-A^i)\,f(h,j-1), \\ F(i-1,j) &= A^{i-1} + (1-A^{i-1})\,f(h,j-1). \end{aligned}$$

These equations, substituted in the first of the fundamental equations, give

$$f(i,j)=A^i(1-b)\,[1-f(h,j,-1)]+f(h,j-1).$$

Let $$C=A^h(1-b),$$

and this equation gives

$$1-f\,(h,j)=(1-C)\,[1-f(h,j-1)]=(1-C)^j$$
$$F(i,j)=A^i+(1-A^i)\,[1-(1-C)^{j-1}]=1-(1-A^i)\,(1-C)^{j-1}$$
$$F(h,\,h)=1-(1-A^h)\;(1-C)^{h-1},$$

which is the probability that A will win at the outset, if he has the cue.

PARTICULAR EXAMPLES.

1. When the player A is an unfailing shot, we have

$$a=1=A,\qquad B=b$$
$$A^h=1$$
$$F(h,\,h)=1.$$

2. When B is the unfailing shot, we have

$$A=a,\qquad b=1=B,$$
$$C=0,\qquad F(h,\,h)=A^h=a^h;$$

so that if (A)'s average run is h, he can afford to give any player whatever a grand discount, if he holds the cue.

3. When A and B are equal players, we have

$$a=b,\qquad A=B=\frac{1}{2-a}.$$

If in this case the average run of each player is a little less than 27 points, or, more accurately, if it is 26.980, either player can venture to give the other a grand discount, if he has the cue, and if the game is 50 points.

4. Other examples for the game of 50 points are contained in the following table: the first column contains the average run of the player (A), who gives the grand discount, and the second column the average run of the player (B). The chances of victory are equal,

if the player who gives the discount holds the cue at the outset. The average run is expressed to thousandths of a point: —

(*A*)'*s* average run.	(*B*)'*s* average run.
42.806	149.513
37.280	72.707
32.963	47.048
29.504	34.074
26.980	26.980
26.975	26.966
26.922	26.834
26.660	26.176
18.948	13.896
13.298	5.459
4.711	1.258

VII.

THE DIMENSIONS AND PROPORTIONS OF THE TEMPLE OF ZEUS AT OLYMPIA.

By Charles Eliot Norton,

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Presented Oct. 10, 1877.

The information given by ancient writers concerning the erection and plan of this famous Temple would seem curiously scanty, but for the fact that there is not one of the most noted buildings of the Greeks, concerning the architecture of which a detailed account has reached us from antiquity. In this instance (as in so many others of the same kind) Pausanias is the chief authority. He tells us that Libon, of Elis, was the architect of the Temple; that it was built from the spoils which the Eleians won from the conquest of Pisa and her neighbors.* This conquest took place in the year 572 B.C.; and it has been inferred, from the words of Pausanias, that the Temple was begun not long after this time. But, if this were the fact, the progress of the work must have been exceedingly slow; for it did not receive the image of the God within its shrine until after the completion of the Parthenon, when Phidias went to Elis for the purpose of making the colossal statue of Zeus; and the figures of the groups in the pediments were not set in their places till about the same time, that is, not far from 435 B.C.†

Pausanias, however, does not say any thing of the date of the erection of the Temple; and no inference can be drawn from the mention of Libon as its architect, for nothing is known of him beyond this statement.‡

The evidence of its date which the plan of the Temple itself affords would lead to the conclusion that it could hardly have been begun before the first quarter of the fifth century B.C.; probably between 480 and 460 B.C.*

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* Pausanias, v. x. 3.

† Müller, "De Phidiæ Vita et Operibus," § 12.

‡ Brunn, "Geschichte der Griechischen Kunstler," ii. 369: "Wir vermögen also nur zu sagen dass er [der Tempel] in der 86. Olympiade vollendet war."

In many points of architectural style, it bears a resemblance to the Temple of Athene (sometimes known as that of Zeus Panhellenius) on the island of Ægina, which was erected in this part of the century; and it stands, perhaps, in still closer architectural relations with the Theseium, at Athens, — a building of which the precise date is not known, but which was probably erected between 470 and 460 B.C.

The Temple of Zeus at Olympia, on account of its national relations, was by far the most important and famous temple of the Peloponnesus. Few temples in the ancient world excelled it in renown, in dignity of association, or in general reverence; few surpassed it in the number and splendor of their dedicatory offerings; few were of larger dimensions, or of more beautiful proportions, or more nobly adorned: —

Οὐκ ἐν ταῖς ζαθέαις Ἀθάναις
Εὐκίονες ἦσαν αὐλαὶ
Θεῶν μόνον.

No other temple, not even that of Delphi, was to an equal degree the temple of the whole Greek race.

Long after Greece had fallen from her station as the living leader of the world in science and in art, long after she had become a Roman dependency, the Olympic Games were still celebrated in sight of this splendid and stately Temple. There is no knowledge of the precise date of its destruction. It was burned probably about the time of the stern edict of the Emperor Theodosius II. against the heathens and their sanctuaries, in the year 426.†

From the time of its destruction, the ruins lay exposed to the injury of flood and storm and earthquake, and to the more wanton injuries of man. By degrees, the soil accumulated over them, and vegetation covered the shapeless heap. The place became solitary and unhealthy;

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* Urlichs, in the "Verhandl. der 25. Versammlung Deutscher Philologen," in Halle, 1867, p. 75, fixes the date of the building between Ol. 77, 3–4, and 80, 3–4; that is, from 470 to 457 B.C. And this view is adopted by Krell, "Geschichte des Dorischen Styls," 1870, p. 85.

† The only notice of its destruction is found in a mutilated note by a scholiast to Lucian's Rhet. Precept, printed in the edition of Jacobitz, Leipsic, 1861, T. iv. p. 221: *διήρκεσεν μέχρι τοῦ μικροῦ . . . ὃς Ἀρκαδὶου υἱὸς ἦν. τ. . . ναοῦ τοῦ Ὀλυμπίου Διὸς ἐμπρη. . .* And so the temple crumbles, and falls to long neglect. Cf. Herkberg, "Der Untergang des Hellanismus," Halle, 1875, p. 428.

and for many centuries no traveller was allured by its ancient fame to visit the scene which had once been the gathering-place of the civilized world. There is an utter blank in its history. But the revived interest in the antiquities and art of Greece, of which the publication of Stuart and Revett's monumental work on the Antiquities of Athens was at once a sign and a stimulus, led travellers, in the early part of this century, to turn their steps to Olympia, in order to investigate the remains which might still be found there.* It was not, however, till 1829 that any thorough exploration of the ruins was undertaken. In that year, they were carefully investigated and measured by an expedition, of which M. Abel Blouet was the head, sent out by the French government; and the results were published, in 1831, in the well-known work, the "Expédition scientifique de Morée." According to Pausanias, the Temple was 230 feet in length, 98 feet in breadth, and 68 feet in height; and it was not without surprise that it was found that the French measures of the length and breadth of the Temple differed widely from the measures given by the only ancient authority.

For more than forty years, no further investigations were undertaken on the spot. But the German expedition which began its work of excavation at Olympia in 1875, and which has already made discoveries of the highest interest and importance, has once more taken the measures of the ruins, and with results generally corresponding with those of the French expedition.†

It thus appears evident that the measures given by Pausanias are incorrect; and this conclusion is confirmed by the fact, that no simple ratio, such as appears to have been frequently, if not always, established in the main dimensions of the Greek temples, exists between the dimensions as stated by him.

It is likely, indeed, that at the time when Pausanias wrote (near the close of the second century of our era), the knowledge of the mathematical principles on which the science of Greek architecture rested, was, if not wholly lost, at least completely neglected. No ancient treatise on the subject has come down to us. Vitruvius gives very little information regarding it; and it has been reserved for students in this century to rediscover some of the elements of this lost science, and to open the way, it may be hoped, to its gradual complete reconstruction.

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* Mr. J. S. Stanhope, in 1824, in his work entitled, "Olympia; or Topography illustrative of the Plain of Olympia, and of the Ruins of the City of Elis," gave a series of views of the general features of the locality, and of the condition of the ruins.

† These measures are given by Herr F. Adler in "Die Ausgrabungen zu Olympia," by Curtius, Adler, and Hirschfeld, Berlin, 1876, p. 20.

It would seem that the architect of a Doric temple began his work by laying out what may be called an ideal scheme, based upon exact numerical relations of the various parts. Having to build a temple of a certain magnitude, he determined the number of columns it should have on the fronts and sides; and fixed the proportions that should exist between the height, the breadth, and the length of his building. These were all to be regulated according to the size of the columns. The height of the column was determined by its diameter at the foot of the shaft; and, as the shaft diminished from base to summit, the upper diameter was accurately proportioned to the lower in a simple ratio. To secure the exactness of these proportions, the diameter of the base was divided into sixty minutes, and these minutes served as a common measure of all the members of the building.

Not merely in the columns, but also in the proportions of the other parts and divisions of their structures, "the Greek architects attached great value to simple ratios of low natural numbers." *

It appears, further, that the ratios employed in any particular building were comparatively few.†

The first point, then, in the investigation of the system of proportions in any given temple, is, after obtaining (if possible) the measure of its breadth and length, to endeavor to ascertain the diameter of the peristyle columns. But here a difficulty exists, unless we have opportunity to measure a number of columns, and so to obtain an average which may afford the probable ideal diameter. For, after having laid out his plan on an exact mathematical scheme, the architect in its execution varied the dimensions of similar parts, — slightly it is true, but still sufficiently to make it unsafe to assume that a single example

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* See "Memoir on the Systems of Proportions employed in the Design of the Doric Temples at Phigaleia and Ægina," by William Watkiss Lloyd (appended to Mr. Cockerell's splendid work on these temples), London, 1860. Mr. Lloyd, in this Memoir, and in a paper "On the General Theory of Proportion in Architectural Design, and its Exemplification in Detail in the Parthenon," in "Papers read at the Royal Institute of British Architects," London, 1859, has made a valuable contribution to the establishment of some of the leading principles of the science of Greek architecture. Among the most important materials for the study as yet provided, are those to be found in Penrose's admirable work on the Parthenon, published by the Dilettanti Society, and in the volume by Mr. Cockerell, of which Mr. Lloyd's Memoir forms a part.

† Lloyd's Memoir, p. 64.

affords the standard, or ideal type. The precision with which the Greek workman — unsurpassed in manual skill, unequalled in the discipline of eye and hand — could reproduce an ideal type, is marvellous in comparison with the work of any other race. But the architect was artist as well as man of science. Numbers were to him what they are to the musician. The harmonies of his art might all be resolved into numerical relations; but these relations, within certain fixed limits, admitted of infinite variety of modulations. There is no dead repetition in a Greek building: each similar member is alike, each is different from the rest. Science gave the law; but within the law there was liberty for the free play of art. A further difficulty in determining the precise typical standard arises from the fact, that, while the refinements of the Greek architecture were almost as far beyond modern perception as they are beyond modern imitation, it is only in the most perfect buildings (such as the Parthenon and the Propylæa) that the execution can be relied upon as answering absolutely to the design. In buildings of a coarser material, — erected under supervision less careful than that of Phidias, and by workmen less disciplined than the Athenian, — the execution often falls short of absolute conformity with what seems to have been the intention; though the defect is generally so slight that it would not be noticeable in other than Greek work.

The measures of the Temple of Zeus at Olympia given by the French and the German expeditions, though not sufficient to enable us to reconstruct every part of the building, are sufficient to enable us to determine something of the system of proportions adopted in its construction; and the study of them leads to some curious results.

The main dimensions as given by the respective expeditions are as follows: * —

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* For the purposes of the investigation, it is necessary to reduce the measures to Olympian or Greek feet. There is still some uncertainty in regard to the precise length of the Greek foot. In the present paper, I assume it, as deduced from a comparison of various ancient measures (see "Smith's Dictionary of Gr. and Rom. Antiquities," Art. *Mensura*), to be in relation to the English foot : : 1 : 1.01125; and hence, 1 metre = 3.24395 Greek feet.

Mr. Lloyd, basing his opinion on the measured length of the front of the Parthenon, makes the ratio of the Greek to the English foot as 1 : 1.01341; and hence, 1 metre = 3.23748 Gr. feet.

M. Aurès in his ingenious "Étude des Dimensions du Grand Temple de Pæstum," Paris, 1868, p. 4, supposes the foot used in the construction of the Parthenon to have measured M. 0.307: hence, 1 M. = 3.25407 Gr. ft. Don Vasquez Queipo, in his "Essai sur les Systèmes métriques et monétaires des anciens Peuples," Paris, 1859, T. i. p. 387, estimates the Greek foot as being about M. 0.30864,

LENGTH ON SURFACE OF STYLOBATE.

French, M. 63.720 × 3.24395 = Gr. ft. 206.70448
German, M. 63.45 × „ = „ 205.82863

BREADTH.

French, M. 27.75 × 3.24395 = Gr. ft. 90.0196
German, M. 27.56 × „ = „ 89.40

The discrepancy shows the difficulty of obtaining exact measurements of the ruins of so large a building, originally constructed of a coarse stone, and so long exposed to the injury of time. Probably, also, the measures were not taken at the same points.

On the French plan, the diameter of a peristyle column is given at 2.244 M.; on the German, at 2.24 M.

2.244 M. = 7.27942 ft.

The French plan alone affords the measure of an upper diameter, 1.696 M. = 5.5017 ft.

Now, as we have no means, by the comparison of a number of measures of different columns, to deduce the ideal diameter, we are obliged to depend on the probable ratio subsisting between the lower and upper diameters. On inspection, it appears that the nearest likely proportion is as three to four. This proportion exists between the

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or 0.308597. Herr Adler ("Ausgrabungen zu Olympia," p. 23) assumes without argument that the Olympian foot was equal to M. 0.3168. This is a wide divergence from the other authorities. The following table shows some of the various estimates: —

Lloyd . . .	1 Gr. ft.	= M. 0.3089;	1 M. =	Gr. ft. 3.23748
Blouet . . .	1 „	= M. 0.3068;	1 M. =	„ 3.25945
Adler . . .	1 „	= M. 0.3168;	1 M. =	„ 3.15656
Aurès . . .	1 „	= M. 0.307;	1 M. =	„ 3.25407
Vasquez Queipo	1 „	= M. 0.3086;	1 M. =	„ 3.24044
Boeckh (deduced from his estimate of the Roman foot, "Public Economy of Athens," trans. by Lamb, p. 127.)	1 Gr. ft.	= M. 0.308211;	1 M. =	„ 3.24453
In this paper .	1 „	= M. 0.308266;	1 M. =	„ 3.24395

diameters of the columns of the Temple at Ægina.* Assuming that this is the true proportion, we correct both diameters, as follows: —

5.5125 : 7.35 : : 3 : 4.

The difference between this hypothetical lower diameter and the measured is something less than 1-104th part of the whole diameter; a difference that, even if we exclude the probability of an original deviation from the standard, may be naturally accounted for by the weathering of 2,300 years.

The French plates further give a measure of the breadth of the abacus, 2.610 M. = 8.4667 ft. This, too, should bear a definite proportion to the diameters; and we find the following ratio: —

5.5125 : 7.35 : 8.575 : : 4.5 : 6 : 7.

The difference between the measured and the computed abacus is considerable, amounting to 11-10ths of an inch, or about 1-84th of the breadth: but it is to be repeated that a single measure is never to be relied upon as giving a standard dimension; while the abacus, thrown down from the capital, has been peculiarly exposed to injury; and its edges may well have been considerably worn.

Dividing the lower diameter by 60, we have the minute or modulus of .1225; of such minutes, the upper diameter contains 45, and the abacus 70.

Having obtained the diameters of the column, we proceed to use them as data for its height. The relation between them is, as has been seen, as 3 : 4, and corresponds with the proportions of the columns of the Temple at Ægina. The height of the latter columns, as measured, is closely equal to five and one quarter times the lower diameter, or to seven times the upper diameter.† Supposing the same proportion to exist here, we have for the height of the column

7.35 × 5.25 = 38.5875.

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* In Plate vii. of Cockerell's work, the diameters of a column of the Æginetan Temple are given as respectively, 3 ft. 3 in. and 2 ft. 5¼ in.; or decimally as 3.25 and 2.438: —

2.4375 : 3.25 : : 3 : 4.

† The height of the Æginetan column (Cockerell, p. 17) is given at 17.19 ft.

Lower diameter 3.25 × 5.25 = 17.0625
Upper „ 2.4375 × 7 = 17.0625

Stated in minutes, the height of the column is 60′ × 5.25 = 315′; and it will be noticed that this is exactly three times the sum of the upper and lower diameters, 45′ + 60′ = 105′ × 3 = 315.′

Having thus obtained a probable height for the column, we have, if possible, to determine that of the whole building. Mr. Lloyd has pointed out the fact, that, in the best architectural works of Greece proper, it was the rule that the height of the column should exceed one-half the height of the building, and that the excess should be equal to one part on a scale by which the whole height of the building was divided into a small uneven number of parts.*

Thus, in the Theseium, the height of the column is 5-9ths of the total height; at Bassæ, the proportion is as 7 : 13; in the Parthenon, as 10 : 19.

Upon trial, it appears that, assuming the height of the columns to be 3-5ths of the height of the building, we obtain a height for the Temple which corresponds proportionately with its measured length and breadth.†

The following table shows the proportions, using the sum of the diameters as a common measure: —

12.8625 × 3 = 38.5875 height of column.
" × 5 = 64.3125 height of Temple.
" × 7 = 90.0375 breadth of Temple.‡
" × 16 = 205.80 length of Temple.

These hypothetical dimensions of breadth and length correspond, as closely as can be required, with the measured dimensions: —

Breadth, as measured by	French	exp.,	90.0196;	hypothetical,	90.0375
" " "	German	"	89.40		
Length " "	French	"	206.70449		
" " "	German	"	205.82863;	hypothetical,	205.80

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* "Expressed more technically, the height of the column compares with the complementary height of the front, as the larger term of a super-particular ratio, — a ratio, that is, of which the terms differ by unity." — Lloyd, Memoir, p. 66.

† This is not far from the proportion of the Theseium; $\frac{5}{9} : \frac{3}{5} :: \frac{25}{45} : \frac{27}{45}$.

‡ The breadth of the abacus is often found to be a measure of the breadth of the temple. In the Theseium, the abacus measures 1-12th of the front; in the Temple at Ægina, it is 1-11th; in the Temple at Corinth, it seems to have been 1-10th; and in the Temple at Olympia, we find it 1-10½.

8.575 × 10.5 = 90.0375.

It seems hardly open to doubt, that we have thus obtained the principal normal measures and proportions of the chief dimensions of the Temple. The result appears still more striking, if we reduce the dimensions to minutes of the diameter: —

$$
\begin{array}{lllr}
7.35 \div 60' = .1225 & \times & 105 = & 12.8625 \\
\text{,,} \qquad \text{,,} & \times & 315 = & 38.5875 \\
\text{,,} \qquad \text{,,} & \times & 525 = & 64.3125 \\
\text{,,} \qquad \text{,,} & \times & 735 = & 90.0375 \\
\text{,,} \qquad \text{,,} & \times & 1680 = & 205.80
\end{array}
$$

The result is confirmed by the close approximation of the ratios to those exhibited in other hexastyle temples of the same period. The ratio of breadth to length, 1 : 2$\frac{2}{7}$, is the same as that of the Theseium; while the ratio of height to breadth is very near to that of the Theseium, the Temple at Bassæ, and the western front of the Propylæa.

Theseum, Bassæ, Propylæa, Height : Breadth :: 3 : 4, or 21 : 28
Temple of Zeus „ : „ :: 5 : 7, or 20 : 28

The measures of length and breadth which we have established are those of the upper lines of the stylobate, or platform upon which the columns stood, and the measure of the height is taken from its surface. The height and breadth of the three steps of the stylobate were carefully proportioned to each other, and to the other dimensions of the Temple; and we have now to determine what their height and breadth were, and thus to ascertain the dimensions of the Temple at the level of the ground.

The breadth of the steps of the stylobate on each end and on each side, as given by the French and German expeditions, is 1.300 M.; but, in this instance as in others, the data are not sufficient to assure us of any thing more than approximate accuracy, 1.300 M. = 4.217 ft.; and this measure, being that of the breadth of the steps on each end or side, is to be doubled to obtain the total addition to length or to breadth.

The height of the three steps of the stylobate as given by Blouet (it is not given by Adler) is apparently 1.546 M. = 5.015 ft.

The hypothetical dimensions for the total breadth of the steps seem to have been 8.575 feet, to be compared with measured
8.434; and, for the height, 5.145,
to be compared with measured 5.015.

If now we add these amounts to the height, breadth, and length of the Temple, on the level of the surface of the stylobate, we have the following dimensions and relations to the common measure: —

H. 64.3125 + 5.145 = 69.4575 = 128625 × 5.40
B. 90.0375 + 8.575 = 98.6125 = „ × 7.666
L. 205.80 + 8.575 = 214.375 = „ × 16.666

The following proportions appear: —

H. from top of styl. : H. from base of styl. :: 25 : 28.
B. at „ : B. at „ :: 21 : 23.
L. „ „ : L. „ „ :: 24 : 25.

The height of the steps on any one side is in the ratio to their breadth of 6 : 5; and it is to the lower diameter of the column in the ratio of 7 : 10; in both these respects corresponding with other examples.*

We have now to ascertain, if possible, the height of the entablature and of the pediment, and their proportions to the other members of the front.

The total height of the Temple from the surface of the stylobate being	64.3125
And the height of the column being	38.5875
We have	25.725

to distribute between the entablature and the pediment.

The only measure of these portions of the building is that of a single block of the architrave given by Blouet (Plate 62), the height of which as measured is 5.4174.

Now, it is found that the height of the total entablature (architrave, frieze, and cornice) is not infrequently in the proportion of one-third of the height of the column, and that the architrave and frieze are of equal or very nearly equal heights, and that the cornice is usually less than one-fifth of the whole height of the entablature.

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* See Lloyd's Memoir, p. 72. In the Temple of Apollo at Bassæ, the proportion of the height of the stylobate to the diameter of the column is as 20 : 30; which may be compared with the 21 : 30 of the Temple of Zeus at Olympia. The proportion of height of steps to their breadth in the Temple of Apollo is as 6 : 5 or 42 : 35; to compare with 42 : 36 of the Temple of Zeus.

In the present case, having no measure of the frieze, we may assume that it was equal in height to the architrave; and we shall find the proportion of the entablature to the column of one-third satisfied, if we assign to the architrave the height of 5.359375, in place of the measured 5.4174.

We have then architrave	5.359375
Frieze	5.359375
Leaving for the cornice	2.14375
To make up the height of the whole entablature	12.8625

or one-third the height of the column. This division of the space — giving 5-12ths to the architrave, 5-12ths to the frieze, and 2-12ths to the cornice — corresponds with close approximation to the divisions of other well-proportioned temples.*

If this be the correct measure of the entablature, the same height remains to be assigned to the pediment; which thus appears to have had the not inappropriate proportion of one-fifth of the whole height.†

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* The entablature of the Parthenon and that of the Theseium are divided as follows: —

Architrave and frieze, each	$\frac{9}{22}$ or	$\frac{108}{264}$
Cornice	$\frac{4}{22}$	$\frac{48}{264}$
While the architrave and frieze of the Temple of Zeus reduced to the same denominator equal, each		$\frac{110}{264}$
And the cornice equals		$\frac{44}{264}$

Other examples are, —

Temple at Bassæ, architrave and frieze each	$= \frac{36}{84}$
In the Temple of Zeus they equal, each	$\frac{35}{84}$
„ Temple at Ægina „ „ „	$\frac{19}{47}$
„ Temple of Zeus „ „ „	$\frac{20}{48}$

† In the Temple at Ægina, the proportion was as 1 : 4.8; in that at Bassæ, as 1 : 4.7. The slight diminution in the proportion in the Temple of Zeus may have been due to its much greater height; the total height of the Æginetan Temple being but about 35.85 ft., and that of the Phigaleian Temple about 36.033.

The total façade seems thus to have had the following vertical dimensions: —

		Pediment,	12.8625	=	105′
Cornice,	5.359375				
Frieze,	5.359375	Entablature,	12.8625	=	105′
Architrave,	2.14375				
		Column,	38.5875	=	315′
			64.3125	=	525′
		Stylobate,	5.145	=	42′
			69.4575	=	567′

The next point to determine is that of the spacing of the columns. It was in the relations of the columns to each other on the fronts or sides of his temple, that the Greek architect found scope for some of the most exquisite rhythms of his art. The distances between them were not to be precisely the same, so as to afford a recurrence of precisely the same optical effects, and to repeat a measure by which the building could be at once divided into so many separate equal parts; but each columniation was to be varied sufficiently to produce the effect upon eyes, so keen and finely disciplined as those of the Greeks, of modulation, and of freedom restrained only by the general law of proportion to which the whole building was subject. On the theoretic plan, they were doubtless laid down with mathematical exactness; in the finished work, each interval had a delicate individuality, which made it incommensurable with the rest. The column at each angle of the building received almost invariably a slightly increased diameter, as having apparently to support a heavier burden than the rest; and, for this and other reasons, the interval between it and the one next to it was less than the ordinary interval.

On the plan in the "Ausgrabungen zu Olympia," the diameter of a corner column is given at M. 2.30, or 7.461 ft. Comparing this measure with the diameter of the average columns, 7.35 ft., the ideal diameter of the angle column appears to have been 7.4725; that is, its diameter at the base was increased by .1225, or one minute of the normal column.

Proceeding now to obtain the sum of the diameters on the front, we have—

4 columns, each 7.35 = 29.40
2 „ „ 7.4725 = 14.945

44.345

Leaving for the five inter-
columnar spaces . . 45.6925

Breadth 90.0375

If the front be divided into five columniations, each extending from the centre of one column to the centre of the next, we have five segments and one diameter to compose the total breadth.

Now, M. Blouet has given measures of four of the columniations on the eastern front of the Temple; so that, by adding to his measures a single diameter, and subtracting the sum from the total breadth, the remainder is the measure of the fifth segment.

The two corner columniations appear to measure, according to Blouet, respectively, 16.09 and 16.0575.[*] The two central columniations measured by him average . . . . 16.8961[†]

The sum of these four columniations is 65.9397

And if we add to this the diameter not included in the
five segments 7.35

We have left for the fifth columniation 16.7478

90.0375

The average of the two corner columniations, as measured, is 16.07375; and the average of the three central columniations is 16.84666.

It seems not improbable that the calculated average of the

2 corner columniations was 16.17 = 132′
3 central „ „ 16.7825 = 137′

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[*] A doubt is occasioned by the fact, that, on the plate in the Expédition scientifique, one of the central columniations is given at M. 4.960, and that this measure differs so widely and abnormally from that of the other central, and agrees so closely with that of the other corner columniation; that I am led to suppose it to be the measure of a corner columniation transferred by oversight to a wrong place on the plan.

[†] Blouet gives only their joint measure.

According to this, the division of the front would have been as follows: —

61′ 71′ 60′ 77′ 60′ 77′ 60′ 77′ 60′ 71′ 61′ = 735′

Feet, 7.4725 8.6975 7.35 9.4325 7.35 9.4325 7.35 9.4325 7.35 8.6975 7.4725 = 90.0375*

In respect to the internal arrangements of the Temple, such measurements as are given by Blouet, and on the German plan, though insufficient for a complete reconstruction, confirm, so far as they go, the conclusions as to the system of proportions adopted in the plan of the Temple which has now been found to exist in its external dimensions.

The breadth of the cella (including under that term the pronaos, the naos, and the opisthodomos) was, —

According to Blouet, M. 15.88 = 51.5139 feet.
" " German plan, 15.86 = 51.45 "

The length of the cella is not given by the Germans, nor in terms by Blouet; but as he gives the measure of the distance from the centre of one of the columns of the pronaos, and also from one of those of the opisthodomos to the edge of the stylobate, it is easy, by subtracting these measures from the measured length of the stylobate, to ascertain that of the cella.

The measure thus obtained † is 147.95 ft.; and the mathematically correct measure appears to have been 147.91875.

The proportion of breadth to length, which, for the whole Temple, is as 1 : 2$\frac{2}{7}$, is for the cella as 1 : 2$\frac{7}{8}$.

Cella, B. 51.45 : L. 147.91875 : : 1 : 2.875.

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* This hypothetical arrangement, though it corresponds very closely to the measured dimensions, is unsatisfactory, in so far as it fails to afford in its main horizontal division any measure equivalent to the height of the column; which, as Mr. Lloyd has pointed out (Memoir, p. 70), was a symmetry frequently at least aimed at by the architect. In the Theseium, the line from the centre of an ordinary column to the further margin of a third was equal to the height of the column. In the Parthenon, three diameters and their two intercolumniations gave the measure. It may be hoped that the German expedition, before the termination of its investigations, will obtain such measurements as may determine, with certainty, the original spacing of the columns on both the fronts and sides of the Temple, and thus afford the means of ascertaining the precise ratios between the solids and the void spaces.

† I subtract from the corrected length of the Temple, — not from that given by Blouet, which (we have seen) is incorrect.

And the length of the cella is to the length of the Temple as 2.875 : 4; while its breadth is to the breadth of the Temple as 4 : 7.

The front of the pronaos and that of the opisthodomos consisted each of two columns between antæ. The diameter of one of these columns is given by Blouet at M. 1.896 = 6.1505 ft., and by Adler as M. 1.89 = 6.131 ft.

The breadth of each of the antæ of the opisthodomos is
given by Blouet as M. 1.78 . . . . = 5.774 ft.
The space between antæ and column . . . . = 8.7714 „
The space between the two columns is, in the pronaos . . . M. 3.128 = 10.1470756 } Av'ge 10.0932692 „
in the opisthodomos, M. 3.064 = 9.9394628 }

Supposing these measures to be correct, the following hypothetical dimensions suggest themselves; but there is no certainty attainable with the existing data.

	Measured.	Hypothetical.		
Breadth of anta,	5.774	5.788125 × 2	= 11.57625 ft.	= 94′.5
Space from anta to column . .	8.7714	8.789375 × 2	= 17.57875 „	= 143′.5
Diam. of column,	6.13	6.125 × 2	= 12.25 „	= 100′.
Intercolumn . .	10.0932692	10.045	= 10.045 „	= 82′.
Total breadth			51.45 „	= 420′.

The breadth between the antæ is 39.87375 ft., which is to the total breadth of the cella as 31 : 40.

The depth of the opisthodomos as measured by Blouet is M. 7.22 = 23.42 ft., — very nearly in the proportion to its breadth, within the walls, of 7 to 12.

23.26 : 39.87 : : 7 : 12.

I have now gone through the principal measurements afforded by the French and German authorities. A few others of less importance will appear in the following tables, which exhibit in a condensed form the results that have been thus far obtained in this investigation, and which present (as it seems to me) an irresistible body of evidence as to the scale adopted for determining the dimensions of the Temple, and as to its principal proportions.*

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* Where two measures are given for the same dimension, the first is the French; the second, the German.

Table of Exterior Dimensions.

	Measured.			Hypothetical.*
Length, on upper step of stylobate,	M. 63.720	=	Gr. ft. 206.78449.	
,, ,, ,,	M. 63.45	=	,, 205.83	205.80
Breadth, ,, ,,	M. 27.75	=	,, 90.0196	90.0375
,, ,, ,,	M. 27.56	=	,, 89.40.	
Height, from ,, ,,				64.3125
Breadth of stylobate	M. 2.60	=	,, 8.434	8.575
Height of stylobate	M. 1.546	=	,, 5.015	5.145
Length at base of stylobate . .	M. 66.05	=	,, 214.264	214.375
Breadth ,, ,, . .	M. 30.35	=	,, 98.4536	98.6125
Height from ,, ,, . .				69.4575
Lower Diameter of peristyle column	M. 2.244	=	,, 7.27945	7.35
,, ,, ,, . . .	M. 2.24	=	,, 7.26645	
Diameter at top of shaft . . .	M. 1.696	=	,, 5.5017	5.5125
Abacus, breadth of	M. 2.610	=	,, 8.4667	8.575
Column, height of				38.5875
Architrave, height of	M. 1.670	=	,, 5.41739	5.359375
Columniations, average of two central	M. 5.2085	=	,, 16.8961	16.84375
Columniations, average of two corner	M. 4.952	=	,, 16.07375	16.078125

Table of Interior Dimensions.

	Measured.			Hypothetical.		
Length of cella, including pronaos, naos, and opisthodomos			Gr. ft. 147.95	147.91875	=	1207.′50
Breadth of cella, including walls	M. 15.88	=	,, 51.5139			
Do. do.	M. 15.86	=	,, 51.45	51.45	=	420.′
Breadth between antæ .	M. 12.32	=	,, 39.9654	39.87375	=	325.′50
Diam. of pronaos column	M. 1.896	=	,, 6.15 }	6.125	=	50.′
,, ,, ,,	M. 1.890	=	,, 6.13 }			

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* The apparent impossibility of conforming the structure to measures so refined as these disappears, if we suppose the masons of the edifice furnished with a rule graduated with minutes of the diameter. Each minute was a little less than one-eighth of a foot. Thus, a measure that looks too delicate for execution — for example, 90.0375 ft. — was simply 735 minutes, and easily to be determined by a rule graduated with ten or five minutes.

TABLE OF INTERIOR DIMENSIONS. — *Continued.*

		Measured.			Hypothetical.	
Breadth of antæ in opisthodomos	M.	1.780	= Gr. ft.	5.774	5.788	= 47./25
Space from antæ to column in opisthodomos	M.	2.700	= „	8.7714	8.789375	= 76./25
Space between columns in pronaos	M.	3.128	10.147	Aver. 10.09	10.045	= 82./
Do. do. in opisthodomos,	M.	3.064	9.939			
Depth of opisthodomos .	M.	7.22	= Gr. ft.	23.42	23.2597	= 189./875
Diameter of naos column,	M.	1.54	= „	4.99568	4.90 or 5.145	= 40./ = 42./
Distance from front of pronaos to edge of stylobate				28.70	28.7875	= 235./
Distance from front of opisthodomos to edge of stylobate				29.15	29.09325	= 237./5
Passage between peristyle and front of pronaos .				21.35	21.4375	= 175./
„ „ „ opisthodomos .				21.80	21.74325	= 177./5
Passage on flank between peristyle and wall of pronaos				11.6364	11.82125	= 96./5

TABLES OF PROPORTIONS.

The following tables show the ratio of measures of various parts to the height, breadth, and length of the Temple: —

H = height. B = breadth. L = length.

Sum of diams.	= 12.8625 ft.	= 105′	× 3	= 38.5875,	H. of col.
„	= „	= „	× 5	= 64.3125	H. of Temple from upper surface of stylobate.
„	= „	= „	× 7	= 90.0375	B. of Temple on upper surface of stylobate.
„	= „	= „	× 16	= 205.80	L. of Temple on upper surface of stylobate.
„	= „	= „	× 5.4	= 69.4575	H. of Temple from base of stylobate.
„	= „	= „	× 7.666	= 98.6125	B. of Temple at base of stylobate.
„	= „	= „	× 16.666	= 214.375	L. of Temple at base of stylobate.

Abacus . .	= 8.575 ft.	= 70./	× 7.50	= H.	From or along surface of Stylobate.
„	= „	= „	× 10.50	= B.	
„	= „	= „	× 24.	= L.	
„	= „	= „	× 8.10	= H.	From or along base of Stylobate.
„	= „	= „	× 11.5	= B.	
„	= „	= „	× 25.	= L.	

B. of Anta .	= 5.788125 ft.	= 47′.25	× 11.111	= H.	From or along surface of Stylobate.
„	= „	= „	× 15.55	= B.	
„	= „	= „	× 35.55	= L.	
„	= „	= „	× 12.	= H.	From or along base of Stylobate.
„	= „	= „	× 17.	= B.	
„	= „	= „	× 37.	= L.	

Architrave .	= 5.359375 ft.	= 43′.75	× 12.	= H.	From or along surface of Stylobate
„	= „	= „	× 16.80	= B.	
„	= „	= „	× 38.40	= L.	
„	= „	= „	× 12.96	= H.	From or along base of Stylobate.
„	= „	= „	× 18.40	= B.	
„	= „	= „	× 40.	= L.	

H. of Stylobate	= 5.145 ft.	= 42′.	× 12.50	= H.	From or along surface of Stylobate.
„	= „	= „	× 17.50	= B.	
„	= „	= „	× 40.	= L.	
„	= „	= „	× 13.50	= H.	From or along base of Stylobate.
„	= „	= „	× 19.166	= B.	
„	= „	= „	× 41.666	= L.	

PROPORTIONS OF CELLA.

B. 51.45 : L. 147.91875 : : 1 : 2.875.
L : L. of Temple : : 2.875 : 4.
B : B. of „ : : 4 : 7.

B. 420′ : B. of T. 735′ : L. of Cella, 1207′.50 : L. of T. 1680′.
: : 1 : 1.75 : 2.875 : 4.

The following table exhibits the mumber of minutes of the lower diameter of the peristyle column, and that of the minutes of the sum of the diameters, and that of one-hundredths of the sum of the diameters contained in the principal dimensions : —

.1225 = 1′ of D.
.128625 = 1-100 of sum of diameters.
.214375 = 1′ of „ „

.1225 : .128625 : .214375 : : 100 : 105 : 175.

		×	×	×
H. of col. . .	38.5875	= 315	300	180.
H. „ . . .	64.3125	= 525	500	300.
B. „ . . .	90.0375	= 735	700	420.
L. „ . . .	205.80	= 1680	1600	960.
H. from base .	69.4575	= 567	540	324.
B. at base . .	98.6125	= 805	766	460.
L. „ . . .	214.375	= 1750	1666	1000.
L. of cella . .	147.91875	= 1207.50	1150	690.
B. „ . . .	51.45	= 420	400	240.

The preceding study of the dimensions and proportions of the Temple had been completed, and the preceding tables drawn up in the form in which they are here presented, when, on a further inspection of them, certain facts appeared, opening the way to unexpected results, which I shall now proceed to set forth briefly.

It will have been noticed that the breadth of the Temple consists of 735 minutes of the diameter of the peristyle column: —

$$.1225 \times 735 = 90.0375\,;$$

and it will be remembered that 7.35 is the measure of the diameter itself. The correspondence is striking; and the probability of its being an undesigned coincidence is diminished, when we further consider that in the ratio between the breadth of the Temple and the height of the column and the height of the Temple, 7 : 3 : 5, the same numbers are repeated.

Now, the lower diameter of the column is the most important dimension of the edifice. It regulates all the others, and its minute supplies the unit of the structure. The choice of 7.35 for its measure, and of 735 for the number of the minutes of which the breadth of the Temple was composed, and of 7, 3, and 5 for the ratio of breadth to heights, seems to indicate that the architect must have had a special motive leading him to select this series of digits to give the law to the proportions of his building.

It is obvious, at first glance, that 735 is composed of the first three *odd* numbers; and that peculiar virtues were supposed by the ancients to be inherent in odd numbers, is a fact familiar to all students. "Numero deus inpare gaudet," from Virgil's Eighth Eclogue (*v.* 76), is a phrase as well known to classical readers as "There 's luck in odd numbers, says Rory O'More," is to another class.

According to Servius, in his comment on this verse, this superstition was "that odd numbers were immortal, because they cannot be divided into two equal parts; the even being mortal." But this is an imperfect explanation of the matter. The superior virtue ascribed to odd numbers lay not merely in their indivisibility, — a quality which was an attribute of the divine nature, but also in the fact that they seemed stronger than the even; for in joining odd and even together, the sum was always odd. Hence, a masculine nature was ascribed to them.*

Three had dignity as the first of the odd numbers, — the first number complete with beginning, middle, and end. By it the triangle, the simplest of planes, was formed; in it were included all bodies under their three dimensions. Five was honored as the first compound of odd and even, and was regarded as thus including the virtues of both in itself. But seven was specially sacred: its properties were so numerous and so remarkable that even Cicero, in the person of Scipio, could say "rerum omnium fere nodus est." †

It might, at first sight, seem that all this subject of the mystical properties of numbers might well be relegated to the domain of merely curious learning. But the ancient doctrine of numbers, as taught by

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* See Plutarch, "Of EI at Apollo's Temple," § 8, Goodwin's edition of "Plutarch's Morals," IV. 485. Also, "Roman Questions," § 25; Id. II. 218. "Cur impares numeros ad omnia vehementiores credimus?" asks Pliny, N. H. xxviii. 5. "Impar numerus mas, par femina vocatur." Macrobius, "Somn. Scip." I. vi. "Numerus impar maribus attributus est." Mart. Capella, "De Nuptiis," &c., II. § 106. "Imparem numerum antiqui prosperiorem hominibus esse crediderunt," Festus, cited by Hardouin in his note on the passage of Pliny quoted above.

† "De Republica," VII. 11. The comment of Macrobius on this passage, "Somn. Scip." I. 6, is full of illustration of the ancient ideas concerning this and other numbers. Of five, he says, "Hic ergo numerus simul omnia et supera et subjecta designat; aut enim deus summus est, aut mens ex eo nata in qua species rerum continentur, aut mundi anima quæ animarum omnium fons est." See, also, Aulus Gellius, "Noct Attic." III. 10. Morhof, in his once-noted "Polyhistor," says, "Quantæ, quam arcanæ numerorum potestates sint, ne in hunc quidem diem satis cognitum est;" and, going on to speak of the special numbers, he says, "Quinarius *τροφὸς*, item *γάμος* dicitur, quod ex binario et ternario, quasi ex femina et mare, conflatus est. Divinus aliis dicitur . . . Septenarium omnium *μυστικώτατον* esse nemo nescit." I. i. xii. 19. In regard to five and seven, see Plutarch, "Of EI," §§ 6–17. "Every number will afford you," says Plutarch, "sufficient matter and argument of praise, if you will but take the pains to look into it; for, to say nothing of others, a whole day would not be enough to express in words all the virtues and properties of the sacred number Seven dedicated to Apollo," § 17.

Pythagoras and his disciples, had more than a merely speculative interest: it had a substantial foundation, and a practical application. For, while in the early progress of the arts numerical relations gave to them their primal and universal laws, the mystical conceptions in respect to the absolute and inherent qualities of numbers themselves quickened the fancy of the artist, and gave him confidence in the performance of his work. "Number and figure were the greatest instruments of thought which were possessed by the Greek philosopher; having the same power over the mind which was exerted by abstract ideas, they were also capable of practical application. . . . They were the measure of all things, and seemed to give law to all things; nature was rescued from chaos and confusion by their power; the notes of music, the motions of the stars, the forms of atoms, the recurrence and evolutions of days, months, years, the military divisions of an army, the civil division of a state,—seemed to afford a present witness of them: what would have become of man, or of the world, if deprived of number?"*

It was not strange that the Pythagoreans, having recognized that the material laws of the universe could be expressed by numbers, should have mistaken this condition for the essence of the thing itself. Such a mistake is frequent in the history of thought; and it may serve to illustrate how superstitious fancies become often mingled with the most solid truths.

But to return to the number which is the special subject of this inquiry, 735, we may observe, that it not only presents in its digits the first three *numeri inpares*, but is itself composed of them, as follows:—

$$7 \times 3 \times 5 \quad \times 7 = 735;$$

and this analysis suggests another and (as I propose to show) highly characteristic and interesting set of factors; namely,—

$$35 \times 21 = 735.$$

In the well-known obscure and much discussed passage in the "Timaeus," in which the Pythagorean Timaeus gives account of the process of making of the soul, we are told that when God had mingled the three elements of which the soul is composed, and out of the three made one, he began to divide into portions the mass he had compounded, in the ratios of 1, 2, 4, 8, and 1, 3, 9, 27; and then proceeded to fill up the intervals between the terms of these ratios with fractional means, so as to form a scale of harmonic numerical proportions. The explanation and interpretation of this scale have occupied commentators both in ancient and modern times.* The exposition by Plutarch, in his treatise, "Concerning the Procreation of the Soul, as discoursed in Timaeus," is of special value as a statement — in great part intelligible, in spite of the corruption of the text — of the ancient conceptions of the nature of the Pythagorean scale. He points out that this Timaean "quaternary,"* as he terms it, contains two series, each commencing with the unit as their common original; next after the unit come, respectively, two and three, the first plane† numbers; then four and nine, the first squares; and, lastly, eight and twenty-seven, the first cubes. The relations of these numbers are exhibited in the following scheme: —

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* Jowett, Introduction to Timaeus, in "The Dialogues of Plato translated," 2d ed. III. 564. The whole passage from which the preceding extract is taken is full of striking and original reflections. Plato's own views in regard to number, and the science of arithmetic as a guide to truth, are set forth in a remarkable passage in his "Republic," book vii. 525, 526.

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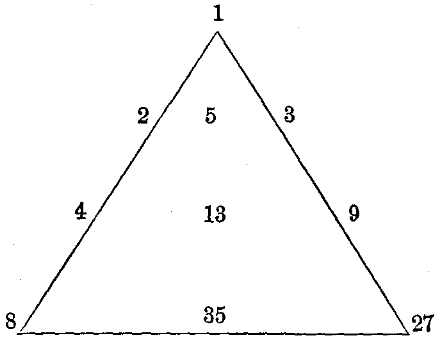
* The fullest and most satisfactory exposition of the remarkable acoustic discovery of Pythagoras in regard to the numerical relations of tones, of the nature of his musical scale, and of the notions based upon it by the ancients, is to be found in Boeckh's treatise, "Ueber die Bildung der Weltseele im Timaeos des Platon," which first appeared in 1807, and is contained in his "Gesammelte Kleine Schriften," Band 3, pp. 109–181, Leipzig, 1866. The subject is more briefly, but well treated of by Westphal, "Harmonik und Melopöie der Griechen" (Metrik der Griechischen Dramatiker und Lyriker, nebst den begleitenden musischen Künsten. Von A. Rossbach und R. Westphal, II. Theil, erste Abtheilung), Leipzig, 1863, pp. 133–139. "Die forschende Geist des Alterthums," says Westphal, "hat wohl über keine wissenschaftliche Entdeckung eine solche Freude gehabt, wie über diesen Fund auf dem Felde der Akustik. In der That macht er dem Alterthum alle Ehre. Die Töne hatten sich als verkörperte Zahlen herausgestellt, die qualitativen Unterschiede waren auf quantitative zurückgeführt. Dies führte zu dem Gedanken dass auch in den übrigen Gebieten des Kosmos in gleicher Weise die Zahl das bestimmende Princip sei. Die moderne Wissenschaft hat durch ihre grossen Entdeckungen in der Chemie und Physik (z. B. in dem chemischen Atomengesetze) die Wahrheit dieses Gedankens gerechtfertigt; aber dem Alterthume war nicht vergönnt, auf diesem Wege weiter zu dringen, man begnügte sich jenen akustischen Zahlen eine absolute Bedeutung zuzuschreiben und sie der ganzen übrigen Welt in einer rein phantastischen Weise zu Grunde zu legen. Die hohe ethische Bedeutung, welche die Musik für das Griechenthum hatte, kann diesen Irrthum entschuldigen, der sogar soweit ging, dass selbst das Seelen-und Geistesleben in jene Zahlenverhältnisse gebannt wurde. Die ganze pythagoreische und platonische Zahlenphilosophie ist auf sie gebaut. Die Zahlen 1, 2, 3, 4 enthielten die drei consonirenden Intervalle (*σύμφωνα*, nämlich 1 : 2 die Octave, 2 : 3 die Quinte, 3 : 4 die Quarte), sie zusammen bildeten den Pythagoreern die Tetraktys. Addirte man die in ihnen enthaltenen Einheiten ($1 + 2 + 3 + 4$) so ergab sich die Zahl 10, und so entstand der Begriff der für die Pythagoreer so bedeutsamen δεκάς. Rechnete man zu jenen Zahlen der consonirenden Intervalle noch die beiden Zahlen 8 und 9, welche das Ganzton-Intervall enthielten, hinzu, so ergab sich $1 + 2 + 3 + 4 + 8 + 9 = 27$; die einzelnen Summanden mitsammt der Summe bildeten hier mit einander 7, und so ergab sich die ἑπτάς. Das sind die sogenannten heiligen Zahlen der Pythagoreer.

Von der zuletzt genannten Heptas geht Plato bei seiner Construction der Weltseele im Timaeus aus. Indem nach ihm der Weltbildner die Weltseele nach diesen Zahlen ordnet,

1, 2, 3, 4, 9, 8, 27



In this manner similar numbers, says Plutarch, are joined together, and they "will produce other remarkable numbers, as well by addition as multiplication. By addition thus: two and three make five, four

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bringt er hiervon zunächst die Zahlen mit einander in Zusammenhang, welche *διπλάσια διαστήματα* (Octaven) und *τριπλάσια διαστήματα* (Duodecimen) bilden.

διπλάσια διαςτ. 1 2 4 8
τριπλάσια διαστ. 1 3 9 27

* This was the second numerical Tetractys of the Pythagoreans. The Tetractys was the root or source of all things, *παγὰν ἀενάου φύσεως* (Carm. Aur. 48.) The first was called the Tetractys of the Ten, and was composed of 1, 2, 3, and 4, the sum of which is the perfect number 10. The second was double, 1, 2, 4, 8 and 1, 3, 9, 27. Each first number represents the point; the second, the line; the third, the plane; the fourth, the solid. The whole Tetractys is 1, 2, 3, 4, 8, 9, 27; the sum of the first six numbers being equal to the seventh. The sacred Seven includes the whole series. See Boeckh, Op. cit. p. 142.

Hierocles, in his comment on the Golden Verses says, *καὶ οὐκ ἔστιν εἰπεῖν ὁ μὴ τῆς Τετρακτύος, ὡς ῥίζης καὶ ἀρχῆς, ἤρτηται.* p. 230; ed. 1654.

† The ancient arithmeticians used the term *plane* numbers for those which were the products of two prime numbers.

and nine make thirteen, eight and twenty-seven make thirty-five. Of all which numbers, the Pythagoreans called five the nourisher — that is to say, the breeder or fosterer — of sound, believing a fifth to be the first of all the intervals of tones which could be sounded. But as for thirteen, they called it the remainder; despairing, as Plato himself did, of being ever able to divide a tone into equal parts. Then, five and thirty they named 'harmony,' as consisting of the two cubes eight and twenty-seven, the first that rise from an even and from an odd number; and as also being composed of the four numbers — six, eight, nine, and twelve * — comprehending both harmonical and arithmetical proportions." †

This comprehensive nature of thirty-five and the various proportional relations of its main factors, admitting of their application to every dimension in a symmetrical system, seem to have induced the architect of the Temple of Zeus to adopt it as the fundamental number for the determination of the dimensions and proportions of the building. For, upon closer investigation, it will be seen that it is not only the chief factor of the measure of the diameter of the column, but that it enters intimately into the determination of the size of every part of the building. For the minute of the diameter, which serves as the universal common measure, is one-sixtieth of 7.35, or .1225; and what is 1225 but 35 × 35? For example of the application of the number to the building, let us take the measure of the breadth in minutes: —

.1225 × 735, that is, as we have just seen, 35 times 35 multiplied by 21 times 35. The abacus is 70′ in breadth, that is 35 × 2; and $35 \times 2 \times 10\frac{1}{2} = 735$, or, as above, $35 \times 21 = 735$.

If we take the length in minutes, a similar result appears: —

.1225 × 1680 = the length; that is, 35 times 35 multiplied by 48 times 35 = the length.

It is further to be noticed that 21, the factor with 35 of the breadth, is the multiple of three times 7; and that 48, the factor with 35 of the length, is composed of 35 and 13, — the latter number holding, according to Plutarch, an important place as "the remainder" in the Pythagorean scale.

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* In the Pythagorean musical scale, 6 denoted the octave, 8 the fifth, 9 the fourth, and 12 unison. See Westphal, p. 133; and Boeckh, § 10, p. 146.

† Macrobius (Somn. Scip. I. vi.), commenting on the passage in "Timaeus," gives a similar scheme of the numbers. He says, "Duos esse primos omnium numerorum cubos, id est, a pari octo, ab impari viginti septem; et esse imparem marem, parem feminam superius expressimus. . . . Coëant enim numeri, mas ille qui memoratur et femina, octo scilicet et viginti septem; pariunt ex se quinque et triginta."

But this is not all: 35 being not only the measure of the minutes, but also of the number of the minutes, of which each of the principal dimensions of the building is composed, is consequently the measure of those dimensions. But here a new subtlety appears. The main dimensions are not simply multiples of 35; but they are multiples of the cube of 35 by the same factor as 35 is of the minutes by which they are measured;* for, the number of minutes which measures each principal dimension being a multiple of 35, and the minute itself being 35^2, the number of feet and decimals of a foot in the dimension equals 35^3, multiplied by the same factor as, multiplied with 35, gives the number of minutes.

Taking the breadth, for an example, again, we have seen that it is stated in

Minutes, thus . . . . $35 \times 21 = 735'$.
Now its measure in feet is
to be stated, thus . . $35^3 \times 21 = 90.0375$
And so with the length . $35 \times 48 = 1680'$.
„ „ . $35^3 \times 48 = 205.8000$

It would seem that the architect, having determined to base his design upon 35, the number called harmony itself, had exercised his ingenuity in devising such dimensions for his Temple as should form a complete and most complex composition of harmonic relations.

He may in this manner have secured that satisfaction for his inventive faculty, and that freedom of independent conception, of which he was deprived, so far as concerned the general character of the building, by the prescribed scheme of the Greek Temple.

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* Cube and square numbers were most highly esteemed by the Pythagoreans. They had discovered that cubes were composed of a regular sequence of odd numbers:—

The first cube, 8 of $3 + 5$.
„ second „ 27 of $7 + 9 + 11$.
„ third „ 64 of $13 + 15 + 17 + 19$, and so on.

In like manner squares, by adding the odd numbers in sequence to unity:—

The first square $4 = 1 + 3$.
„ second „ $9 = 1 + 3 + 5$.
„ third „ $16 = 1 + 3 + 5 + 7$.
„ fourth „ $25 = 1 + 3 + 5 + 7 + 9$, and so on. See Theonis Smyrnaei, Expositio eorum quæ in Arithmeticis ad Platonis Lectionem utilia sunt. Ed. Gelder, Leyden, 1827, c. xv. (p. 43); c. xxv. (p. 63).

The following table exhibits the composition of the principal dimensions:—

	Minutes.		Feet.	
Diameter at base	60′.	=	7.35	= 35 × 21.
,, at top	45′.	=	5.5125	= 35 × 14.
Sum of diameters	105′.	= 35 × 3	12.8625	= 35^3 × 3.
Height of column	315′.	= 35 × 9	38.5875	= 35^3 × 9.
,, of temple	525′.	= 35 × 15	64.3125	= 35^3 × 15.
Breadth of ,,	735′.	= 35 × 21	90.0375	= 35^3 × 21.
Length ,,	1680′.	= 35 × 48	205.8000	= 35^3 × 48.
Breadth at base of stylobate	805′.	= 35 × 23	98.6125	= 35^3 × 23.
Length ,, ,,	1750′.	= 35 × 50	214.3750	= 35^3 × 50.
Length of cella	1207′.5	= 35 × 34.5	147.91875	= 35^3 × 34.5
Breadth ,,	420′.	= 35 × 12	51.45	= 35^3 × 12.*

I am not aware that the application of the Pythagorean scale in any of the works of Greek architecture has heretofore been observed. It is not improbable that the architect of the Temple at Olympia may have made an exceptional use of the numbers of musical harmony. He may have been a Pythagorean by training. But it also is not unlikely (although we have no evidence as yet of the fact) that the Greek architects recognized a relation between the harmonies of music and those of their own art; and that some of them, adopting the Pythagorean doctrine of numbers as the key of Nature, and the origin of the order of the universe, believed that the most perfect works of their art were to be achieved by the conformity of their designs to the principles of the architecture of the world. The law of harmony must be one for the rolling of the spheres, and for every tone of the lyre or the lute; one for the proportions of the starry habitations, and for those of the earthly temples of the gods.

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* It will be noticed that the factors of the dimensions are, except in two instances, whole multiples of 7, or 3, or 5, and that, of the two exceptions, one, 23, is composed of 3 multiplied by 7.666, and the other, 34½, of 3 multiplied by 11.5.

VIII.

ON THE PHOTOGRAPHIC ACTION OF DRY SILVER BROMIDE COLLODION, &c., TO RAYS OF SOLAR LIGHT OF DIFFERENT REFRANGIBILITY.

By Robert Amory, M.D., *Longwood*.

Presented June 10th, 1877.

1. In continuation of experiments reported to the Academy in May, 1876, I present the following results: Further experiments demonstrate that silver bromide collodion is sensitive to all the visible rays of the spectrum, but requires a very prolonged exposure to the less refrangible rays. A gradual decrease of photographic action in reducing the silver salt is observed from about line F to line A, the latter being almost imperceptible, though by careful observation a very thin deposit of metallic silver can be made out at the red end.

2. Some of the glucosides, such as salicin, amygdalin, populin, and santonin, slightly increase the photographic action of the silver salt, when exposed to the less refrangible rays. This reduction of the silver salt does not occur as a selective action, — that is to say, in certain isolated portions of the spectrum; but the photographic image is somewhat stronger at F, and gradually fades out towards line A.

3. Stewart-Wortley dry plates, which contain salicin and uranium with the silver bromide, on exposure to the solar spectrum, show an increased action from F to A, and the image is stronger than when a glucoside alone is added to the silver bromide; but in these there is no selective action below line F.

4. Colonel Abney claims that the addition of benzoin and bromine to a collodion made at a high temperature increases the sensitiveness of a silver bromide dry plate to the less refrangible rays of the spectrum, and to some of the ultra red rays. Only one plate out of a large number prepared according to Colonel Abney's directions* bears in my hands any confirmation of his results. It should, however, be observed that the presence of benzoin and bromine in a sensitized collodion plate,

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* The details of the process were kindly sent by him to me.

on account of a more minute subdivision of the reduced silver, gives a better-defined and sharp image.

5. Silver bromide in gelatine (gelatino-bromide emulsion) shows much more action to all rays of the visible spectrum than when this salt is suspended in collodion: the strength of the image gradually diminishes from the blue to the red end, but gives no selective action below line F.

6. Some of the aniline dyes dissolved in a collodion sensitized with silver bromide containing an excess of silver nitrate and then dried, give in my hands very positive results, which, in detail, are as follows: It is well known that solar light passing through a solution of magenta red, fuchsin, water-blue, or œosin, all being aniline colors, shows selective absorption of certain rays of the solar spectrum. If a silver bromide collodion be faintly colored with œosin, for instance, then immersed in a strong bath of silver nitrate, so that there shall be over the surface of the collodion an excess of silver nitrate, which after immersion should be thoroughly washed off, and dried in the dark; then if this be exposed to the image of the spectrum in focus, a stronger image of the solar lines will be developed upon the collodion between E and D lines, the maximum of action being between these lines; and a faint image will be developed between the F and E lines. In other words, there is a selective photographic action in that portion of the spectrum (viz., between E and D) where the absorption band of the sunlight colored by this aniline dye is usually observed. Attention is specially called to the fact, that, if the pigment is added to the sensitized collodion after the excess of silver nitrate has been washed off, the effects above mentioned do not follow. We have taken advantage of this phenomenon to photograph quite distinct images of solar lines between F and B, which were refracted by two dense glass prisms* and projected by a common 40-in. spectacle lens. I exhibit these photographs to the Academy. I also exhibit the photographic image of the absorption bands of blood, of didymium nitrate and potassic permanganate in solution; likewise, a solution of grass chlorophyll. This latter substance offers unusual difficulties, from the fact that, in order to get chlorophyll absorption bands, the sunlight must be very strongly colored with chlorophyll, which circumstance allows an amount of light insufficient to effect the silver reduction, unless under a prolonged exposure. This plate was exposed one and a half hours;

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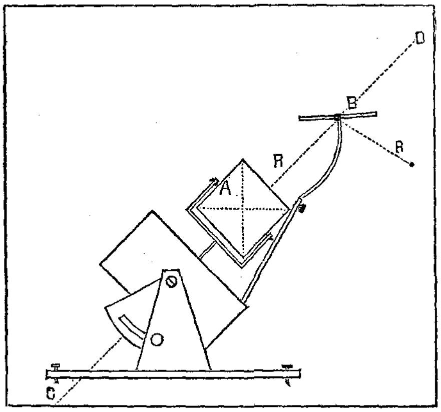
* These prisms were made by Adam Hilger of London, being duplicates of those he made for Colonel Abney for his photographic experiments.

consequently the image is somewhat blurred, both from forced development as well as from diffused light. The scientific value, then, of these phenomena consist in enabling us to photograph an image of that portion of the spectrum which comprises rays of less degree of refrangibility than line F, and especially those between lines E and C, and perhaps also at B. It is in this part of the spectrum that many natural colors absorb the rays of light. All these photographs of absorption bands exhibited to the Academy were taken by the dried silver bromide collodion colored by œosin and fuchsin.

We devised for these experiments a new form of heliostat, the beam of sunlight reflected from which varied so slightly that we were able to expose some of our plates for an hour and a half without any re-adjustment of the instrument. The cheap cost of this form of heliostat and its simple mechanism, compared with other instruments of equal accuracy, induce me to present its description to the Academy.



As will readily be seen, this instrument consists of a stand, equatorially mounted, and containing a small watch movement, in the train of which a wheel is connected, which revolves once in twenty-four hours. In the centre of this wheel a pinion is fixed, upon which a mirror can be adjusted and firmly set. When placed in its correct position, the reflection of a central beam of sunlight is thrown directly in a line C D parallel to the polar axis of the earth. An upright rod, carrying another silvered mirror B, intercepts at any angle to the first mirror this central beam of light, and reflects it in any desired position. In our experiments, the path of this beam is horizontal, and passes through a hole in a dark shutter upon the slit of the spectroscope. We used the reflection from the silver surface of the mirror, as affording us more light.

The advantages of this instrument lie in its simple construction and the ease of its accurate adjustment. The angle of declination in relation to the North star on that part of the earth in which the heliostat is to be used being found, the mirror A is adjusted to throw its central ray R of reflected sunlight upon the centre of the mirror B; consequently, as the mirror B is fixed in the axis of the instrument, the beam of light must necessarily travel along that axis, and the error caused by the motion of the earth is thus compensated as accurately as if the angle of declination were calculated upon the arc of the circle whose radius is equal to a line drawn from the centre of the surface of mirror A to that of mirror B.

In the course of my experiments, the following observation has been noted: —

It has generally been supposed that the human eye is not sensitive to that part of the solar spectrum which contains rays of a higher degree of refrangibility than H H. To show that this phenomenon is not constant, the following observation is offered: If the solar spectrum be projected by a converging lens, and received upon the retina of the eye at its proper visual focus, the *extra*-violet rays and their solar lines as far out as L are visibly distinct.

To make this phenomenon more apparent, let the image of the spectrum be received upon an opaque paper screen, and from which a hole has been cut out, so that only that portion of the spectrum between H and L may pass through and be received upon the retina of the eye. A little practice in receiving the image at its proper focus will produce the effect above alluded to.

IX.

ON THE LONGITUDE OF WALTHAM, MASS.

By Leonard Waldo,
Assistant at the Observatory of Harvard College.

Presented Nov. 14, 1877.

At the request of the Mechanical Superintendent of the American Watch Company's factory in Waltham, Mass., the longitude of their private observatory has been determined as nearly as may be from one night's exchange of signals; and, as that observatory is near a Coast-Survey station, the result has sufficient value to be placed on record for any investigation relative to the problem of station error.

The manner of determining the longitude was as follows: In order to eliminate, as far as possible, the errors in the resulting longitude arising from the lack of a simultaneous action among all the armatures of the electro-magnets used in transmitting and recording the clock-beats at the two stations, it was arranged that both observers should use the same clock, and should, as far as possible, have the same manner of connections at both observatories. The mean-time Bond clock, used for time signals at 97 Water Street, was therefore employed. This clock, working through the private wires operated by the time-service of the observatory, and by the American Watch Company, recorded its beats simultaneously at Waltham and at Cambridge. At both these stations, local circuits included a chronograph, a relay worked by the Boston clock, and the observing key of the observer.

The observations were made at Cambridge with the broken-telescope transit, by M. Herbst of Poulkova. This is the instrument ordinarily used for time observations, and may be found figured and described in vol. viii. of the "Annals of Harvard College Observatory." It has a clear aperture of 2.75 inches, and a focal length of 32.68 inches, nearly. The pivots are of steel, 1.195 inches in diameter, and sensibly equal. They rest upon V-shaped gun-metal bearings, which are $0^{\text{in}}.16$ in breadth, and whose centres are distant from each other 19.22 inches. The horizontal axis consists of two reversed cones, which terminate in a cube whose single edge is 4.43 inches. The instrument is provided with three positive eye-pieces, magnifying, respectively, 36, 67, and 99 diameters; this last being the one employed in the longitude observations. The reticule of the instrument consists of a series of twenty-five lines ruled by Prof. W. A. Rogers. They are arranged in tallies of five lines each; the equatorial interval between the lines being $3^{s}.543$, and between the middle lines of the separate tallies $21^{s}.258$. This form of reticule is found extremely convenient in practice; since a circumpolar star may be observed over the five lines of a single tally, and the instrument can be easily reversed in time to observe a symmetrical tally with the circle opposite its first position. The equal spacing of the lines possible, when they are ruled on glass, reduces by one step the computation of the results. And, by a slight movement of the eye-piece, it is possible in this reticule to obtain a spurious image of a line remarkable for its blackness, without sensibly disturbing the focal image of the star. This is especially convenient in observing faint stars, where it becomes necessary to reduce the illumination so much that the line would ordinarily be invisible. On the other hand, the loss of light by reflection is objectionable; and the writer has not been able to use lines on glass with bright-line illumination for faint stars.

The chronograph by Bond & Sons, situated in the computing room of the observatory, was used.

At Waltham, the observations were made with a transit instrument by Alvan Clark & Sons. It is of the ordinary pattern, has a clear aperture of 2.55 inches, and a focal length of 38.0 inches nearly.

The pivots, 0.972 inches in diameter, rest upon journal bearings 1.00 inch in length. These bearings are distant from each other 17.25 inches. The telescope is provided with a diagonal eye-piece magnifying 64 diameters, and it is reversed by means of a reversing carriage.

The chronograph used is also from the shop of Alvan Clark & Sons; it is of the cylinder pattern, and is controlled by the conical pendulum governor so successfully used by the Messrs. Clark in this connection.

For portable instruments, the formula expressing the relation between the apparent place of a star and its observed place, as affected by errors of observation, may be written

$$\alpha = T + \tau + \Delta\tau + \delta T\,(T - T_0) + A\,a\ (\text{or } A'\,a') + B\,b + C\,c - 0^{s}.021 \cos\varphi \sec\delta,$$

where

α is the adopted right ascension of the star.

T is the observed time of transit.

τ is an approximate correction to the time-piece employed.

$\varDelta\tau$ is the correction to τ.

δT is the hourly rate of the time-piece.

$T - T_0$ is the interval in hours between the time of observation and the mean of the times of observation of all the stars combined in one group.

$A\,a$ is the correction for error of azimuth, when the Circle is East.

$A'a'$ is the correction for error of azimuth, when the Circle is West.

$B\,b$ is the correction for error of level.

$C\,c$ is the correction for error of collimation, positive when the Circle is East.

$0^s.021 \cos\varphi \sec\delta$ is the correction for diurnal aberration.

Where the hourly rate is extremely small, and the error of the time-piece is determined at the same instant at both stations, we may write

$$\alpha - (T + B\,b + \tau - 0^s.021 \cos\varphi \sec\delta) = \varDelta\tau + A\,a \text{ (or } A'\,a') + C\,c;$$

and, putting for the first member the known term γ, the equation becomes

$$0 = -\gamma + \varDelta\tau + A\,a \text{ (or } A'\,a') + C\,c$$

for each star observed.

In the case of a fixed instrument, we should have

$$0 = -\gamma + \varDelta\tau + A\,a + C\,c.$$

The normal equations for the first case are, —

$$\begin{aligned}
0 &= -\Sigma\gamma + \Sigma\varDelta\tau + \Sigma A\,a + \Sigma A'\,a' + \Sigma C\,c,\\
0 &= -\Sigma A\,\gamma + \Sigma A\,\varDelta\tau + \Sigma A^2\,a + \Sigma A\,A'a' + \Sigma A\,C\,c,\\
0 &= -\Sigma A'\,\gamma + \Sigma A'\,\varDelta\tau + \Sigma A'\,A\,a + \Sigma A'^2\,a' + \Sigma A'\,C\,c,\\
0 &= -\Sigma C\gamma + \Sigma C\,\varDelta\tau + \Sigma C\,A\,a + \Sigma CA'a' + \Sigma C^2\,c;
\end{aligned}$$

and for the second case,

$$\begin{aligned}
0 &= -\Sigma\gamma + \Sigma\varDelta\tau + \Sigma A\,a + \Sigma C\,c,\\
0 &= -\Sigma A\,\gamma + \Sigma A\,\varDelta\tau + \Sigma A^2\,a + \Sigma A\,C\,c,\\
0 &= -\Sigma C\gamma + \Sigma C\,\varDelta\tau + \Sigma CA\,a + \Sigma C^2\,c.
\end{aligned}$$

Adopting the following values of τ, and using the star places given in the catalogue of 529 stars issued by the Astronomischen Gesellschaft, we have the data, —

For Harvard College Observatory, $\tau = - 16^{s}.260$;
For Waltham, Mass., $\tau = - 42.855$;

1877, October, $17^{d}.4$.

Star.	Adopted R. A.	Observed Transit. Harvard College. $b = + 0^{s}.054$.	Observed Transit. Waltham. $b = - 0^{s}.048$.
	h. m. s.	h. m. s.	h. m. s.
γ Sagittæ	19 53 19.71	. . . .	6 7 53.39
θ Aquilæ	20 5 0.52	6 19 4.48	6 19 32.50
ο Cygni	20 9 47.17	6 23 50.22	. . . .
κ Cephei	20 12 57.25	6 27 59.84	6 27 24.41
γ Cygni	20 17 50.87	6 31 52.66	6 32 20.18
ε Delphini	20 27 38.37	6 41 23.36	6 41 51.12
73 Draconis	20 32 65.58	6 47 4.74	6 47 30.83
ε Aquarii	20 41 4.43	6 55 3.04	. . . .
76 Draconis	20 51 18.86	7 2 18.62	7 5 43.73
ξ Cygni	21 0 29.67	7 14 25.56	7 14 52.87
77 Draco	21 7 54.89	7 21 50.80	7 22 16.62
α Cephei.	21 15 40.17	7 29 33.64	7 30 1.09
β Aquarii	21 25 8.55	. . . .	7 39 28.15
74 Cygni	21 31 63.78	. . . .	7 46 22.05

and the following values of γ: —

Star.	δ	Harvard College Observatory.		Waltham, Mass.	
		Position of Circle.	γ	Position of Circle.	γ
γ Sagittæ	+ 19°.2	. . . . .	. . . . .	West	— $0^{s}.26$
θ Aquilæ	— 1.2	East	+ $0^{s}.93$	West	— 0.47
ο Cygni	+ 46.4	East	+ 1.02	. . . . .	. . . . .
κ Cephei	+ 77.3	East	+ 0.68	West	+ 3.21
γ Cygni	+ 39.9	East	+ 0.96	West	+ 0.13
ε Delphini . . .	+ 10.9	East	+ 0.89	West	— 0.22
73 Draconis . . .	+ 74.5	East	+ 1.00	West	+ 0.98
ε Aquarii	— 9.9	West	+ 0.37	. . . . .	. . . . .
76 Draconis . . .	+ 82.1	West	— 2.23	East	— 0.50
ξ Cygni	+ 43.4	West	— 0.04	East	— 0.70
77 Draco	+ 77.6	West	— 1.49	East	— 0.24
α Cephei	+ 62.1	West	— 0.23	East	— 0.86
β Aquarii	— 6.1	. . . . .	. . . . .	East	— 1.18
74 Cygni	+ 39.9	. . . . .	. . . . .	East	— 0.93

For the stars observed at Harvard College, we have the following equations, where the coefficients are computed with the value of the latitude: —

$$\varphi = +\ 42^\circ\ 22'.8.$$

For Circle East, —

$$\begin{aligned}
0 &= -\ 0.93 + \Delta\tau + 0.69\ a + 0.00\ a' + 1.00\ c,\\
0 &= -\ 1.02 + \Delta\tau - 0.10\ a + 1.45\ c,\\
0 &= -\ 0.68 + \Delta\tau - 2.61\ a + 4.56\ c,\\
0 &= -\ 0.96 + \Delta\tau + 0.06\ a + 1.31\ c,\\
0 &= -\ 0.89 + \Delta\tau + 0.55\ a + 1.00\ c,\\
0 &= -\ 1.00 + \Delta\tau - 1.99\ a + 3.74\ c;
\end{aligned}$$

and for Circle West, —

$$\begin{aligned}
0 &= -\ 0.37 + \Delta\tau + 0.00\ a + 0.81\ a' - 1.02\ c,\\
0 &= +\ 2.23 + \Delta\tau - 4.64\ a' - 7.26\ c,\\
0 &= +\ 0.04 + \Delta\tau - 0.02\ a' - 1.38\ c,\\
0 &= +\ 1.49 + \Delta\tau - 2.70\ a' - 4.67\ c,\\
0 &= +\ 0.23 + \Delta\tau - 0.71\ a' - 2.13\ c.
\end{aligned}$$

A discussion of these equations has shown that the introduction of the term $A'\ a'$ gives values of a and a' sensibly the same.

We may therefore dispense with this term, and our normal equations become

$$\begin{aligned}
0 &= -\ \ 0.86 + 11.00\ \Delta\tau - 10.66\ a - \ \ 3.40\ c,\\
0 &= -\ 12.06 - 10.66\ \Delta\tau + 41.53\ a + \ \ 40.74\ c,\\
0 &= -\ 34.69 - \ \ 3.40\ \Delta\tau + 40.74\ a + 122.60\ c;
\end{aligned}$$

which give

$$\begin{aligned}
\Delta\tau &= +\ 0.395,\\
a &= +\ 0.156,\\
c &= +\ 0.242.
\end{aligned}$$

But it is better to substitute the values of $a\ a'$ and c in the equations for the time stars, and thus determine a value for $\Delta\ \tau$. We have then, for Circle East, —

for θ Aquarii, the equation $0 = -\ 0.93 + \Delta\tau + .107 + .242,$
„ o Cygni, „ $0 = -\ 1.02 + \Delta\tau + .016 + .350,$
„ γ Cygni, „ $0 = -\ 0.96 + \Delta\tau + .009 + .317,$
„ ε Delphini, „ $0 = -\ 0.89 + \Delta\tau + .085 + .242;$

and for Circle West, —

for ε Aquarii, the equation $0 = -0.37 + \varDelta\tau + .126 - .246$,
„ ξ Cygni, „ $0 = +0.04 + \varDelta\tau - .003 - .333$, [rej]
„ α Cephei, „ $0 = +0.23 + \varDelta\tau - .110 - .515$.

From which we derive

$$\varDelta t = +0^{s}.562 \pm 0^{s}.026,$$

and the resulting clock correction

$$\begin{aligned} \tau + \varDelta\tau &= -16^{s}.260 + 0^{s}.562 \pm 0.026, \\ &= -15^{s}.698 \pm 0^{s}.026; \end{aligned}$$

whence the Bond clock is fast of Harvard College Observatory, sidereal time, —

$$15^{s}.698 \pm 0^{s}.026,$$

as determined by L. Waldo.

The character of the mounting of the transit instrument at Waltham, combined with a previous discussion of the observations, renders the introduction of the term $A' a'$ superfluous. And in the absence of a determination of the latitude, the latitude is assumed to be

$$\varphi = 42^{\circ}\ 23'.$$

From the observed stars, we derive the following equations: —

for Circle West, $0 = +0^{s}.26 + \varDelta\tau + 0.42\ a + 1.06\ c$,
$0 = +0.47 + \varDelta\tau + 0.69\ a + 1.00\ c$,
$0 = -3.21 + \varDelta\tau - 2.61\ a + 4.56\ c$,
$0 = -0.13 + \varDelta\tau + 0.06\ a + 1.31\ c$,
$0 = +0.22 + \varDelta\tau + 0.55\ a + 1.02\ c$,
$0 = -0.98 + \varDelta\tau - 1.99\ a + 3.74\ c$;

and for Circle East, $0 = +0.50 + \varDelta\tau - 4.64\ a - 7.26\ c$,
$0 = +0.70 + \varDelta\tau - 0.02\ a - 1.38\ c$,
$0 = +0.24 + \varDelta\tau - 2.70\ a - 4.67\ c$,
$0 = +0.86 + \varDelta\tau - 0.71\ a - 2.13\ c$,
$0 = +1.18 + \varDelta\tau + 0.76\ a - 1.01\ c$,
$0 = +0.93 + \varDelta\tau + 0.05\ a - 1.30\ c$;

from which we derive the normals: —

$$
\begin{aligned}
0 &= \quad 0.00 + 12.00\,\Delta\tau + 10.14\,a + 5.06\,c,\\
0 &= + 8.22 - 10.14\,\Delta\tau + 41.62\,a + 18.07\,c,\\
0 &= - 37.56 - 5.06\,\Delta\tau + 18.07\,a + 122.88\,c.
\end{aligned}
$$

The solution of which gives

$$
\begin{aligned}
\Delta\tau &= -\,0.175,\\
a &= -\,0.395,\\
c &= +\,0.357, \text{ for Circle East.}
\end{aligned}
$$

But, deriving our final value of $\Delta\tau$ from the time stars only, we have, —

$$
\begin{aligned}
\text{for Circle East, } 0 &= +\,0^s.26 + \Delta\tau - 0^s.17 + 0^s.38,\\
0 &= +\,0\,.47 + \Delta\tau - 0\,.27 + 0\,.36,\\
0 &= -\,0\,.13 + \Delta\tau - 0\,.02 + 0\,.47,\\
0 &= +\,0\,.22 + \Delta\tau - 0\,.22 + 0\,.36;
\end{aligned}
$$

$$
\begin{aligned}
\text{and for Circle West, } 0 &= +\,0\,.70 + \Delta\tau + 0\,.01 - 0\,.49,\\
0 &= +\,0\,.86 + \Delta\tau + 0\,.28 - 0\,.76,\\
0 &= +\,1\,.18 + \Delta\tau - 0\,.30 - 0\,.36,\\
0 &= +\,0\,.93 + \Delta\tau - 0\,.02 - 0\,.46.
\end{aligned}
$$

From which we derive

$$
\begin{aligned}
\Delta\tau &= -\,0^s.402 \pm 0^s.031,\\
\text{and } \tau + \Delta\tau &= -\,42.855 - 0^s.402 \pm 0^s.031;
\end{aligned}
$$

whence the Bond clock is fast of the sidereal time at the Waltham Station

$$43^s.257 \pm 0^s.031,$$

as determined by C. V. Woerd.

From observations made of the stars ξ Lacertæ, δ Cephei, and ι Cephei, at Waltham, on the evening of Oct. 22, 1877, it was found that on that evening L. Waldo observed an equatorial star $0^s.269$ before C. V. Woerd observed the same star.

The Russian transit pier, at Harvard College Observatory, is 44.5 feet West of the centre of the dome, to which longitudes are usually referred.

If now we correct the observed error of the clock at Harvard College Observatory by the difference of the personal equations of the observers given above, and reduce the position to the centre of the large dome, we have for the assumed error of the clock, as determined at Harvard College Observatory, —

$$+ 15^{s}.698 \pm 0.026 + 0^{s}.269 - 0^{s}.039,$$

which is

$$+ 15^{s}.928 \pm 0^{s}.026;$$

and for Waltham we have

$$+ 43.257 \pm 0.031.$$

And their difference gives

$$- 27^{s}.329 \pm 0^{s}.040.$$

The most recent determination of the longitude of the centre of the dome of the Harvard College Observatory determines it to be east of Washington

$$0^{h}\ 23^{m}\ 41^{s}.11;$$

whence our final result is that the transit pier in the private observatory erected on Crescent Street, by the American Watch Company, at Waltham, Mass., is East of the centre of the dome of the United States Naval Observatory at Washington, D. C.

$$0^{h}\ 23^{m}\ 13^{s}.78 \pm 0^{s}.04.$$

Harvard College Observatory, November, 1877.

X.

THE MOON'S ZODIACAL LIGHT.

By L. Trouvelot.

Presented Nov. 14, 1877.

During the evening of April 3, 1874, the "zodiacal light" was particularly brilliant; especially close to the horizon, where it appeared as a segment of a circle, having an irregular wavy outline, giving it a vague resemblance to the beams of a faint aurora. Although the sky was clear, it was found impossible to observe with the telescope on that night, on account of the great disturbance of the atmosphere. At 9^h 45^m, the declination needle indicated a very strong magnetic perturbation in Cambridge, oscillating through an angle of 3° 22′. However, no aurora was visible at this time, although the phenomena usually attending them were manifested during the evening by the tremulous appearance of the telescopic images.

While going home, I remarked in the East a strange conical light rising obliquely from the top of the roof of a building, behind which the moon, then about 15° or 20° above the horizon, was concealed from view. By going away from the building, the conical light, which closely resembled the tail of a comet, became brighter and brighter as it approached the moon, upon the western limb of which it rested. The base was at least as wide as the diameter of the moon; but it extended beyond, on each side, by a fainter light, which gradually vanished in the sky. The extension of this luminous appendage I estimated to be equal to eight or ten times the moon's diameter. It was not readily visible when the moon was in sight, as the brilliant light of our satellite overpowered its dim brilliancy. The axis of this appendage was found to be coincident, or nearly so, with the ecliptic; and its line prolonged in the west passed a little to the north of Jupiter.

The phenomenon had been observed for about fifteen minutes, when it gradually faded away until it almost totally disappeared five minutes later, although the sky was clear. A quarter of an hour after, the sky was overcast with dense vapors, which continued for nearly an hour.

At $11^h\ 0^m$ the sky had cleared up, and the moon shone brightly. The luminous appendage was still visible, and even appeared more brilliant than before. In order to ascertain whether this appendage was visible only on one side of the moon, or if it was seen on the other side, I went under the piazza of my house, and placed myself in such a position as to have the moon concealed by its upper part, the sky below being visible. As I expected, a similar appendage was observed on the eastern side of the moon, exactly opposite the western one; the axis of both wings, passing through the moon's centre, being in the plane of the ecliptic.

Although at this moment no auroral light was seen in the north, yet up in the zenith there were evident signs of it, as luminous vapors assembled there and rapidly dissolved, arranging themselves into bands radiating from a centre after the manner of the crown of bright auroras. At $11^h\ 20^m$, all traces of the luminous vapors in the zenith had vanished; and at the same time the appendages of the moon were almost totally invisible, although the sky remained clear.

The fact that the zodiacal light had been unusually brilliant during this evening, and that the two luminous appendages of the moon resembled it in shape and appearance, and were situated in the same plane, seems to indicate that the two phenomena are of the same order; while the magnetic perturbation and the auroral phenomena connected with the variation of brightness observed in the moon's appendages would seem to indicate some kind of connection between the zodiacal light and the aurora. The result of my observations of the zodiacal light and the aurora during the last seven years also seems to indicate some such connection; as, when the zodiacal light was observed to be particularly bright, it has generally been followed by auroral phenomena. But only a long series of observations in this direction can solve the problem.

Cambridge, Nov. 2, 1877.

XI.

UNDULATIONS OBSERVED IN THE TAIL OF COGGIA'S COMET, 1874.

By L. Trouvelot.

Presented Nov. 14, 1877.

On the evening of July 21, 1874, at $9^h\ 0^m$, the moon being in her first quarter, and the sky remarkably clear even close to the horizon, my attention was attracted by a bright ray of light darting from the north-western horizon, way up in the constellations. Taking it for an auroral phenomenon, I went in for the spectroscope; but on my return, after a few seconds, to my disappointment I found no more trace of it. Soon, however, it reappeared, and darted up in an instant after the manner of certain auroral rays, and vanished again after ten or fifteen seconds. I then became aware of my error, and found with surprise that the phenomenon was taking place in the tail of Coggia's comet, the head of which was then plunged under the horizon.

During the whole time that I observed this interesting phenomenon, I saw the comet's tail shortening and extending, lightening up and extinguishing like the rays of certain auroras. Extended undulations, rapid vibrations, ran along it in succession from the horizon to its extremity, giving it the appearance of a fine gauze wavering in a strong breeze. The pulsations and the waves of light were of unequal duration; some being rapid, while others lasted a longer time. For over one hour, the comet's tail kindled and extinguished more than one hundred times; the extinction being sometimes so complete that it was impossible to see any trace of the comet; while sometimes it became so bright that, in spite of the light of the moon, it could be distinguished easily in all its contours, even to its very extremity, which was then a little to the south of γ Ursa Majoris.

Be it coincidence or not, at the moment that this phenomenon was occurring, a strong magnetic perturbation was going on in Cambridge, where the declination needle oscillated through an angle of 1° 27′, although no auroral light was seen; and by the kindness of Mr. Cleveland Abbe, of the Signal Corps, I learn that no aurora was reported for that night.

It is not a new thing to see vibrations and pulsations running along the tails of comets. Many observers have seen this phenomenon; among others, Longomontanus, Vandelin, Snellius, and Father Cysat, who are reported to have seen undulations taking place in the border of the comet of 1618, as if it was agitated by the wind. Hevelius observed analogous motions in the comets of 1652 and 1661. Pingré asserted that he distinctly saw, in the long tail of the comet of 1769, "des ondulations semblables à celles que les aurores boréales présent-ent." * According to Winnecke, from the 5th to the 12th of October, 1858, the rays forming the superior part of Donati's comet spread and contracted suddenly, like the rays of the aurora.

CAMBRIDGE, Jan. 5, 1877.

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* Arago, Astro. Popu., vol. ii. p. 439, Paris, 1855.

XII

SUDDEN EXTINCTION OF THE LIGHT OF A SOLAR PROTUBERANCE.

By L. Trouvelot.

Presented Nov. 14, 1877.

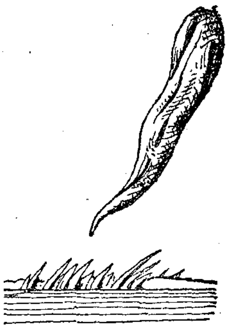

FIG. 1

On the 26th of June, 1874, while making my daily observation of the sun with the spectroscope at the Harvard College Observatory, I saw an unusual phenomenon, which may be worth recording. The narrow slit of the instrument was directed on the preceding side, about 270°, just above a group of spots which was then very near the limb, when I saw a brilliant protuberance partly projected on the spectrum, on the side of the rays of less refrangibility. In shape, this hydrogen flame resembled an elongated comma, having its acute extremity directed towards the sun, where it terminated just a little above the chromosphere. The chromosphere under this protuberance formed several slender and acute aigrette-shaped flames, none of which, however, reached it. The large prominence, which was slightly inclined to the limb, had a height of $3' 37''$, and about 3° in its greatest width. Fig. 1.

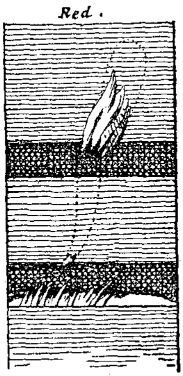

FIG. 2

When the slit was set wide open, so as to allow the whole protuberance to be seen between its jaws, the comma-shaped flame appeared perfect, and showed plainly its texture. But, when it was observed with a narrower slit, it became partly invisible on the C line; only a short and jagged portion being seen in it on the red side. Fig. 2. When the slit was carried along the protuberance by means of its screw, the portion visible on the C line did not remain constant, but either extended or contracted of a small quantity; the maximum portion visible on the C line never being more than one-fourth the width of the slit, while sometimes it was not seen at all on this line.

The portion of the protuberance projected on the spectrum was considerably more brilliant than the spectrum itself, and about one-third only of its whole length was visible. As the slit was carried along it, the visible parts became invisible near the C line, and invisible parts appeared on the spectrum; and the area of the visible portion either contracted or extended, when seen in different parts.

I had been observing this phenomenon for eight or ten minutes, when, while, looking at it with the slit wide open, the flame suddenly vanished, at 10^h 30^m, no traces of it remaining. As no motion of any kind, no extension, no contraction, could be perceived before or at the moment this phenomenon took place, and as the light did not go out of it gradually, but as suddenly as a flash of lightning, it does not seem that a change of position was the cause of its disappearance, but rather because the light which rendered it visible abandoned it in an instant.

According to theory, this protuberance was moving rapidly away from the earth at the moment of the observation, as it was projected upon the less refrangible side of the spectrum; yet this would fail to explain its sudden disappearance, since for this it should have moved out of sight with an inconceivable velocity.

For over half an hour I watched attentively the same spot in expectation of seeing the flame reappear; employing for this a narrow and a wide slit in succession, but with entirely negative results. I saw no more traces of it, although the small aigrette-shaped flames of the chromosphere, which were still visible, indicated the exact place where it had vanished, and where very probably it still existed, but now as a dark protuberance.

On several occasions I have seen the light abandon a protuberance gradually, but never so suddenly and on such a grand scale; and sometimes I have seen also the light gradually illuminating protuberances which were invisible before, something after the manner of clouds in our atmosphere lighting up and fading into darkness by the appearance or disappearance of the sun. Of course, the illumination of dark solar protuberances cannot be conceived as being due to the reflection of light, as in the case of the clouds in our atmosphere: it is the protuberance itself which is rendered luminous by some change taking place in it. These observations would seem to indicate that on the sun there are sometimes dark and non-luminous protuberances, which may cause the spots of absorption often observed in the vicinity of spots.

The phenomenon of the gradual illumination of a protuberance was observed in 1869, at Des Moines, Iowa, during the total eclipse of the sun, by Professor William A. Rogers, who accompanied Dr. C. H. F. Peters, on the Litchfield Eclipse Expedition. Professor Rogers was observing a large protuberance on the sun with a 9-inch-aperture refractor, when he saw several protuberances form gradually in the vicinity of the large flame, and at a considerable height above the chromosphere.

The projection of the hydrogen flames on the spectrum is not a very rare phenomenon during the period of maximum of sun spots, and it has been observed several times. However, it may be worth while to record a characteristic case of projection, accompanied with remarkable changes of form, and a visible motion of the protuberance.

On Sept. 10, 1872, at $12^h\ 33^m$, I was observing a small narrow flame forming an arch on the chromosphere, the height of which was equal to 36″. Fig. 3. Nothing remarkable was to be seen in this protuberance, although it was in the vicinity of a group of spots which was then very near the eastern limb of the sun; but, two minutes later, one of the extremities of the arch reposing on the chromosphere was suddenly detached from the limb, springing up like a distended bow, ascending in an instant to a height of 70″, then appearing straight and rigid, but twisted like a rope. Fig. 4. For a few seconds, it continued to ascend, at the same time growing wider; and at $12^h\ 37^m$, it had attained its maximum altitude of 118″. It was then slightly curved. Fig. 5. At $12^h\ 43^m$, the force which had carried it up began to give way, and it then descended rapidly towards the sun, folding

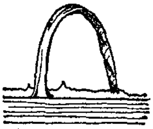

FIG. 3

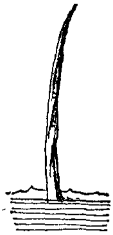

FIG. 4

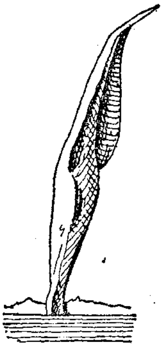

FIG. 5

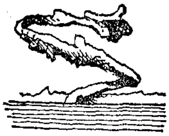

FIG. 6

upon itself in two places, while at the same time it became narrower. Fig. 6. At 12^{h} 45^{m}, it had reached its former height; and soon after, it sunk to a level with the chromosphere, and was lost in it.

At the same instant that the arc of hydrogen was distended, it was seen projected on the spectrum opposite the sun, towards the violet. The figure of this protuberance appeared exactly the same, whether it was projected on the spectrum or seen between the wide-open jaws of the slit. However, when the slit was narrow, the flame became invisible on the C line, although it remained projected on the spectrum. When the protuberance, after having reached its greatest altitude, descended rapidly towards the sun, it remained projected on the spectrum just as before, although the descending motion was apparently in a contrary direction to the ascending one; but this did not seem to affect the position of the flame on the spectrum.

Cambridge, Jan. 12, 1877.

XIII.

ON SATURN'S RINGS.

By L. Trouvelot.

Presented Nov. 14th, 1877.

In No. 2146 of the "Astronomische Nachrichten," Professor Asaph Hall, in giving the results of his observations on the planet Saturn, makes some remarks on my observations of the same planet which were published in the Proceedings of the American Academy for the year 1875–1876.

Professor Hall began his observations in June, 1875. "At first," he says, "my attention was not specially given to the appearance of the Ring. . . . After the picture of Saturn was made by Mr. L. Trouvelot with our telescope in September, 1875, I gave more attention to the appearance of the Ring, and I have done so during the last year. . . . On account of the confidence I have in the drawings made by so skilful an artist, I have been surprised to find that I have never been able to see the slightest trace of two phenomena of the Ring which Mr. Trouvelot draws with the greatest distinctness."

Here Professor Hall refers: first, to the notch which I have represented in the shadow of the globe of Saturn on the Ring; second, to the jagged or tooth-like appearance of the principal division on the ansæ.

Had Professor Hall consulted his memory, undoubtedly his surprise would have been less; since he would have remembered that on the same drawing to which he refers, and which I made at the Naval Observatory in his presence, I represented the shadow of the Ball with its convexity turned towards the planet, just as he saw it later and described it, and as indeed I continued to see it during 1876. Knowing that the shadow was curved inward in September and not notched, I fail to understand why he should have expected to see a notched shadow rather than a curved one, when almost a year had elapsed between his observations and mine, in December, 1874.

Besides myself, Schröter, Lassell, De La Rue, Jacob, Bond, Coolidge, Tuttle, and many others, have seen the shadow more or less notched.

From these observations, it would seem that this phenomenon is not a very rare one; but it is not permanent, as Professor Hall appears to have supposed.

I am indebted to Professor Edward S. Holden, of the Naval Observatory, for an interesting drawing and observation of Saturn, which he made with the 28-inch silvered-glass reflector of Dr. Henry Draper of New York, on the night of Sept. 8, 1874. At his request, Dr. Draper has kindly sent to me a tracing of his original drawing, accompanied with the memorandum recorded in the note-book, at the moment of the observation. It reads as follows: "Observation of Sept. 8, 1874. Division of rings seen all round; inner ring greatly brighter than outer, particularly on the outer edge of it: main belt *triple*, reddish brown in color; upper and lower edges of belt sharp. Shadow of ball, on ring, like this; *i.e.*, funnel-shaped." Fig. 1.

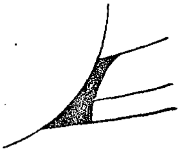

Fig. 1.

As to the jagged appearance of the outer border of the principal division, Professor Hall has seen no trace of it. He says: "The only approach to the appearance of the division as drawn by Mr. Trouvelot that I have ever seen has been when the image of the planet was tremulous, and the sky so clear as to give a distinct but unsteady view of the division of the Ring. At such times the unsteady appearance of the division might lead to some such view as that given by Mr. Trouvelot; but still I think he must have seen something quite different." After saying that, during six or eight nights in a year, their large telescope gives excellent images of Saturn, he continues: "On these nights the appearance of the planet is very beautiful; but my experience is that on these rare nights one will see fewer strange phenomena about the Ring and the shadows than when the images are blurred and indistinct."

Even if I could have been so greatly deceived as to represent for realities the deformations undergone by images in rapid vibrations, I am pretty certain that I could not have seen the delicate dark angular forms which I have represented, but rather rounded, ill-defined forms totally wanting in the blackness and sharpness of those which I saw. Contrary to Professor Hall's suggestion, it is precisely when the definition was the most perfect that the "strange phenomena" could be seen with more distinctness, and at the moment the image became tremulous in the least, it disappeared confounded with the dark division of the rings.

The fact that Professor Hall has not been able to see the "Pencil line," even during one of these beautiful nights he speaks of, sufficiently indicates his failure to see the jagged appearances of the principal division; and indeed, he could not have expected to see it, as these forms are almost as difficult to make out as the grayish line of the outer ring.

I have no positive evidence that these markings continued visible after the end of September, 1875; as, after that time, I discontinued somewhat my observations on Saturn, looking at it only occasionally, until the present year, during which, I have observed it on every possible occasion. But, of course, the obliquity of the Ring is too great now to allow the observation of such delicate forms, although I still continue to see the principal division on the ansæ. It is not impossible that the obliquity of the Ring was the cause of the failure of Professor Hall to verify my observations, or the phenomenon may be a temporary one, and it may have been absent when he made his observations.

The phenomenon of the jagged border of the principal division, as I have represented it, was seen so often and with such distinctness in 1872, when the Ring was wide open, that it was impossible for me to doubt its reality; and, besides, it was verified at least on two occasions by Professor Winlock, the late director of the Harvard Observatory, who once was accompanied by Mr. Miliken, manager of the Western Union Telegraph Company, who also saw the dark angular forms on the following ansa.

Professor Hall seems to be in doubt as to the reality of the anomalous curvature of the shadow of the planet on the Ring, and appears inclined to attribute this appearance to some illusion caused by the varied conditions of our atmosphere. In reply, I will remark that, if such was the case, how could we explain its long duration as concave, and its no shorter duration as convex, which has been alternatively observed since the time of Cassini?

Several years ago, a very distinguished and industrious observer, F. Angelo Secchi of the Roman Observatory, pointed out that the deformation of the shadow of the Ball on the Ring was the natural consequence of the unevenness of the surface receiving it. If this is the true explanation, as I think it is, the natural consequence, as derived from the observations, is that the form of the surface is not permanent, since the shadow has evidently shown different outlines; appearing at different times either as a straight, a convex, a concave, or a notched line.

Cambridge, Oct. 5, 1877.

XIV.

SUPPLEMENTARY NOTE ON THE THEORY OF THE HORIZONTAL PHOTOHELIOGRAPH.

BY PROFESSOR WILLIAM HARKNESS, U. S. NAVY.

Presented Dec. 12, 1877.

REFERRING to equation (14),* we remark that, if the photographs of the sun are centred upon the measuring engine with moderate care, it will seldom happen that either δy or δx amounts to so much as 3″, while A will usually exceed 900″. It is therefore evident that all three of these quantities cannot be accurately derived from conditional equations of the same form as (14), without using logarithms having at least six places of decimals. As the value of A is not required, we eliminate it in the following manner: In equation (14) put

$$\left.\begin{aligned} \sin\,(\varepsilon \mp \theta) &= a \\ n\cos\,(\varepsilon \mp \theta) &= b \\ \tfrac{1}{2}R\,[\sin^2(\varepsilon \mp \theta) + n\cos^2(\varepsilon \mp \theta)] - \frac{nA^2}{2\,R} &= n \end{aligned}\right\} \qquad (58)$$

Then the resulting normal equations will be

$$\left.\begin{aligned} 0 &= [an] + [aa]\,\delta y + [ab]\,\delta x \\ 0 &= [bn] + [ab]\,\delta y + [bb]\,\delta x \end{aligned}\right\} \qquad (59)$$

If m is the number of equidistant points at which R is measured, the expressions for the required auxiliaries are

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* Proceedings of the American Academy of Arts and Sciences, Vol. XII. (1876–77), p. 194.

$$\left.\begin{aligned}
[aa] &= \Sigma \sin^2 (\varepsilon \mp \theta) = \tfrac{1}{2}\, m \\
[ab] &= n\, \Sigma \sin (\varepsilon \mp \theta) \cos (\varepsilon \mp \theta) = 0 \\
[bb] &= n^2\, \Sigma \cos^2 (\varepsilon \mp \theta) = \tfrac{1}{2}\, m\, n^2 \\
[an] &= \Sigma \sin (\varepsilon \mp \theta) \left\{ \tfrac{1}{2} R [\sin^2 (\varepsilon \mp \theta) + n \cos^2 (\varepsilon \mp \theta)] - \frac{nA^2}{2\,R} \right\} \\
[bn] &= n\, \Sigma \cos (\varepsilon \mp \theta) \left\{ \tfrac{1}{2} R\, [\sin^2 (\varepsilon \mp \theta)] + n \cos^2 (\varepsilon \mp \theta)] - \frac{nA^2}{2\,R} \right\}
\end{aligned}\right\} \quad (60)$$

As the value of $[ab]$ comes out zero, we now evidently have

$$-\delta y = \frac{[an]}{[aa]} \qquad -\delta x = \frac{[bn]}{[bb]} \qquad (61)$$

If m is an even number, there will be $\frac{1}{2}\, m$ pairs of values of the angle $\varepsilon \mp \theta$, and in every case the two angles which compose the pair will differ from each other by 180°. If R and R' are the radii vectores of the two points composing a pair, and if the summation is extended only through $\frac{1}{2}\, m$ points, then putting

$$D = \tfrac{1}{2} (R - R') \left[1 + \frac{nA^2}{RR'} + (n - 1) \cos^2 (\varepsilon \mp \theta)\right] \qquad (62)$$

we have

$$\left.\begin{aligned}
-\delta y &= \frac{2}{m} \sum_{\epsilon \mp \theta = \pi}^{\epsilon \mp \theta = 0} \sin (\varepsilon \mp \theta)\, D \\
-\delta x &= \frac{2}{mn} \sum_{\epsilon \mp \theta = \pi}^{\epsilon \mp \theta = 0} \cos (\varepsilon \mp \theta)\, D
\end{aligned}\right\} \qquad (63)$$

To simplify these expressions still further, we remark that n is always very near unity, not becoming so small as 0.950 until a zenith distance of 85° is reached. Assuming $n = 1$ is equivalent to supposing the apparent sun to be replaced by an artificial one of perfectly circular outline, whose area is the same as, and whose centre coincides with the centre of gravity of, the apparent sun. As this assumption does not affect the values of δy and δx, we adopt it; and then equation (62) reduces to

$$D = (R - R') \qquad (64)$$

As the assumed artificial sun is circular in form, it has no minor axis; and therefore θ may be put equal to zero, and we have finally

$$\left.\begin{aligned} -\delta y &= \frac{2}{m} \sum_{\epsilon=\pi}^{\epsilon=0} \sin \varepsilon \, (R - R') \\ -\delta x &= \frac{2}{m} \sum_{\epsilon=\pi}^{\epsilon=0} \cos \varepsilon \, (R - R') \end{aligned}\right\} \qquad (65)$$

from which the values of δy and δx can be found by means of a table of natural sines and cosines, and a Crelle's multiplication table; no logarithms being required.

As it is now evident that the assumption of an elliptical outline for the apparent sun is unnecessary, we proceed to determine for δr a more accurate expression than equation (16). With this view, let *abcd*, Fig. 5, be the outline of the true sun, and *i* its centre, through which the vertical circle *ac* passes; and let *ebfd* be the outline of the apparent sun, and *k* its centre of gravity. Then, if *gh* is any diameter of the

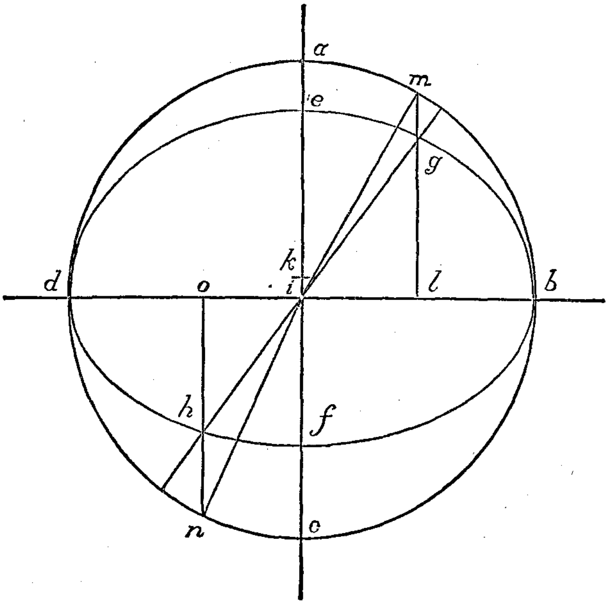

Fig. 5.

true sun, taking *ac* and *db* as a pair of rectangular axes, and putting $ai = s$, $gl = y_1$, $ho = y_2$, $aig = \theta$, $aim = \theta_1$, $ain = \theta_2$, we have

$$\left.\begin{aligned} y_1 &= s \cos \theta_1 - s \cos \theta_1 \frac{dr_1}{d\zeta_1} \\ y_2 &= s \cos \theta_2 - s \cos \theta_2 \frac{dr_2}{d\zeta_2} \end{aligned}\right\} . \tag{66}$$

in which $\frac{dr_1}{d\zeta_1}$ is the first derivative of the refraction, taken at a point midway between *m* and *l*; and $\frac{dr_2}{d\zeta_2}$ is the same derivative taken at a point midway between *n* and *o*. Adding these two equations, we obtain

$$y_1 + y_2 = s (\cos \theta_1 + \cos \theta_2) - s \cos \theta_1 \frac{dr_1}{d\zeta_1} - s \cos \theta_2 \frac{dr_2}{d\zeta_2} \tag{67}$$

and, as the factors for converting the cosines of θ_1 and θ_2 into the cosine of θ can never differ much from unity, it will be quite accurate to write

$$y_1 + y_2 = s (\cos \theta_1 + \cos \theta_2) + s^2 \cos^2 \theta \frac{d^2 r}{d\zeta^2} \tag{68}$$

in which $\frac{d^2 r}{d\zeta^2}$ is the second derivative of the refraction, taken at the centre of the true sun.

If we put $il = x_1$, and bear in mind that on account of refraction the horizontal diameter of the sun is contracted by a constant ratio, μ, then we have

$$x_1 = \mu s \sin \theta_1 \tag{69}$$

and

$$\tan \theta = \frac{x_1}{y_1} = \tan \theta_1 \frac{\mu}{1 - \frac{dr_1}{d\zeta_1}} \tag{70}$$

or

$$\tan \theta_1 = \frac{\tan \theta}{\mu} \left(1 - \frac{dr_1}{d\zeta_1}\right) \tag{71}$$

Regarding θ_1 and $\frac{dr_1}{d\zeta_1}$ as variable, and differentiating, we get

$$\frac{d\,(\tan \theta_1)}{d\zeta} = - \frac{\tan \theta}{\mu} \frac{d^2 r_1}{d\zeta^2_1} \tag{72}$$

Assuming the well-known relations,

$$\frac{d\,(\cos\theta_1)}{d\theta_1} = -\sin\theta_1 \qquad (73)$$

$$\frac{d\,(\tan\theta_1)}{d\theta_1} = \sec^2\theta_1 \qquad (74)$$

dividing (73) by (74), multiplying the quotient by (72), and writing θ for θ_1, we have, with sufficient accuracy,

$$\frac{d\,(\cos\theta_1)}{d\zeta} = \left[\frac{\sin^2\theta\cos\theta}{\mu}\right]\frac{d^2r}{d\zeta^2} \qquad (75)$$

The vertical distance between the centres of the ordinates y_1 and y_2 being very approximately $s\cos\theta$, we now have

$$s\,(\cos\theta_1 + \cos\theta_2) = -\left[\frac{\sin^2\theta\cos^2\theta}{\mu}\right]s^2\,\frac{d^2r}{d\zeta^2} \qquad (76)$$

and therefore

$$y_1 + y_2 = \left[\cos^2\theta - \frac{\sin^2\theta\cos^2\theta}{\mu}\right]s^2\,\frac{d^2r}{d\zeta^2} \qquad (77)$$

The value of μ is 0.99974. Consequently, it may be taken equal to unity without appreciable error, and then

$$y_1 + y_2 = \cos^4\theta\, s^2\,\frac{d^2r}{d\zeta^2} \qquad (78)$$

As y_1 and y_2 are any pair of opposite ordinates of the curve which defines the outline of the apparent sun; and as δr is the distance *ik*, or, in other words, the ordinate of the centre of gravity of the apparent sun; we have, by reasoning identical with that employed in deducing equation (63),

$$\delta r = s^2\,\frac{d^2r}{d\zeta^2}\cdot\frac{2}{m}\sum_{\theta=+\frac{1}{2}\pi}^{\theta=-\frac{1}{2}\pi}\cos^4\theta \qquad (79)$$

in which m is the number of points on the sun's limb at which ordinates have been computed. If we assume these points to be equidistant upon the apparent sun, and take $m = 12$, as in the measurements of the photographs, then

$$\delta r = 0.375\, s^2\,\frac{d^2r}{d\zeta^2} \qquad (80)$$

To test the amount of inaccuracy introduced in equation (79), by writing θ for θ_1 in (75), and omitting μ in (78), the rigorous expression for δr, involving θ, θ_1, and μ, was formed; and, upon substituting in it the numerical values of these quantities for the extreme case where the sun has a zenith distance of 80°, the numerical coefficient in (80) became 0.374. This differs so little from the value before found as to lead to the conclusion, that equation (79) is probably as accurate as the refraction tables themselves.

We have thus found the values of δy, δx, and δr, and it only remains to deduce from them the expressions for $R \sin \varepsilon$ and $R \cos \varepsilon$; R and ε being respectively the radius vector and angle of the image of the true sun's centre, referred to the original system of polar co-ordinates given by the measuring engine. An inspection of Fig. 4 shows that δr is measured along the line $S'_* v'$, while δy and δx are measured along lines respectively parallel to, and perpendicular to, $S'_* Z$. Hence, as the angle ε is measured from a line at right angles to $S'_* Z$, and θ_s is the angle $ZS'_* S'$,

$$\left.\begin{aligned} R \sin \varepsilon &= \delta y + \delta r \cos \theta_s \\ R \cos \varepsilon &= \delta x \mp \delta r \sin \theta_s \end{aligned}\right\} \qquad (81)$$

in which δr must be regarded as essentially negative, θ_s must always be taken less than ninety degrees, and the double sign must be understood in the same way as in the equations (10). It is, perhaps, scarcely necessary to add, that these are the values of $R \sin \varepsilon$ and $R \cos \varepsilon$, which must be employed in the equations (18).

To facilitate the computation of δr from equation (80), it is desirable to have a table giving the numerical values of the second derivative of the refraction. To form such a table, we assume Bessel's expression for the refraction; namely,

$$r = \alpha \tan \zeta \qquad (82)$$

where r is the refraction, expressed in seconds of arc, and α is a quantity which varies slowly with the zenith distance, ζ. The first derivative of this expression is

$$\frac{dr}{d\zeta} = \frac{1}{\cos^2 \zeta}\left[\alpha + \tan \zeta \cos^2 \zeta \frac{d\alpha}{d\zeta}\right] \qquad (83)$$

which becomes

$$\frac{dr}{d\zeta} = \frac{1}{\cos^2 \zeta}\,\alpha' \qquad (84)$$

by putting

$$\alpha' = \alpha \text{ arc } 1'' + \tan \zeta \cos^2 \zeta \frac{d\alpha}{d\zeta} \tag{85}$$

Differentiating (84), we get

$$\frac{d^2 r}{d\zeta^2} = \frac{2 \tan \zeta}{\cos^2 \zeta} \left[\alpha' + \tfrac{1}{2} \cot \zeta \frac{d\alpha'}{d\zeta}\right] \tag{86}$$

which becomes

$$\frac{d^2 r}{d\zeta^2} = \frac{2 \tan \zeta}{\cos^2 \zeta} \alpha'' \tag{87}$$

by putting

$$\alpha'' = \alpha' \text{ arc } 1'' + \tfrac{1}{2} \cot \zeta \frac{d\alpha'}{d\zeta} \tag{88}$$

The factor, arc 1″, is introduced in equations (85) and (88), because in (82) α is expressed in seconds of arc, while α' and α'' must be expressed in parts of radius. Bessel has given a table of the values of α', which may be found in his "Astronomische Untersuchungen," Vol. I. pp. 198 and 199; and also in Chauvenet's "Spherical and Practical Astronomy," Vol. II. pp. 572 and 573; but it must be noted that our α' is Bessel's α''. From the differences of consecutive values of α' given in that table, the values of $\frac{d\alpha'}{d\zeta}$ have been obtained, and then the values of α'', given below, were computed by means of equation (88).

The quantities in the following table correspond to a state of the atmosphere such that the barometer would stand at 29.597 inches, its attached thermometer at 32° F., and the external thermometer at 48°.75 F. The first and fourth columns of the table contain the argument. The second and fifth columns contain the values of log. α'', to the argument true zenith distance, computed by means of equation (88); but it must be carefully noted that, in using these logarithms, their characteristics must be diminished by 10. The third and sixth columns contain the values of δr, to the argument true zenith distance, computed by means of equation (80); and the seventh column, headed $\delta r'$, contains the values of δr to the argument apparent zenith distance, for all cases where they differ sensibly from the values given in the sixth column.

Strictly speaking, α'', δr, and $\delta r'$ are functions, not only of the zenith distance, but also of the density of the atmosphere; but so long as the temperature of the latter remains between + 30° and + 75° F., and its pressure between 29 and 30 inches of mercury, the values of δr and $\delta r'$ given in the table will not be in error more than $0''.01$, for any zenith distance less than 80°. For temperatures, pressures, or zenith distances, beyond these limits, if great accuracy is desired, the values of α'', δr, and $\delta r'$, must be multiplied by the factor $\beta^{A''}\gamma^{\lambda''}$, in which $\beta = BT$, and the values of A'', λ'', B, T, and γ, are to be taken from Bessel's table, cited above. To prevent misapprehension, it may be well to note that the values of A'' and λ'' given by Bessel apply rigorously only to the first derivative of the refraction; but their use with the second derivative gives rise to so little error that it does not seem worth while to recompute them especially for it.

ζ	Log. α''	δr	ζ	Log. α''	δr	$\delta r'$
°		″	°		″	″
0	1.1314	0.000	52	1.1238	0.003	
4	.1289	.000	54	.1229	.004	
8	.1286	.000	56	.1218	.004	
10	.1286	.000	58	.1205	.005	
12	.1286	.000	60	.1190	.006	
16	1.1286	0.000	62	1.1173	0.008	
20	.1285	.000	64	.1147	.010	
24	.1284	.001	66	.1118	.012	
28	.1283	.001	68	.1080	.016	
30	.1282	.001	70	.1034	.021	
32	1.1281	0.001	72	1.0972	0.029	0.029
34	.1280	.001	74	.0887	.040	.040
36	.1279	.001	76	.0776	.058	.059
38	.1278	.001	78	.0605	.089	.090
40	.1274	.001	80	.0310	.144	.147
42	1.1270	0.002	81	1.0107	0.188	0.193
44	.1265	.002	82	0.9858	.253	.261
46	.1259	.002	83	.9504	.349	.363
48	.1252	.002	84	.8991	.491	.517
50	.1246	.003	85	0.8431	0.747	0.809

Washington, May 30, 1877.

XV.

CONTRIBUTIONS FROM THE CHEMICAL LABORATORY OF HARVARD COLLEGE.

RESEARCHES ON THE SUBSTITUTED BENZYL COMPOUNDS.

By C. Loring Jackson.

THIRD PAPER.

ON PARAIODBENZYL COMPOUNDS.

C. F. Mabery.

Presented May 9, 1877.

The preparation of *paraiodbenzylbromide*, $C_6H_4ICH_2Br$, has been described in the first article of this series;* but the method there given did not invariably yield good results, the product often being mixed with a black tar which it was very hard to remove completely by crystallization. A more careful study of the conditions of the reaction has shown that this was due to an oily impurity of the iodtoluol, a very small amount of which was sufficient to destroy a great part or even the whole of the paraiodbenzylbromide: if the paraiodtoluol purified by distillation was carefully pressed between filter-paper, this oil was absorbed, and then the product of the bromiring was nearly white after the first crystallization. When the purified paraiodtoluol was used, the yield was very nearly the same, averaging 4.7 grs. of pure bromide from 10 grs. of iodtoluol, when the bromiring took place at 115°, 130°, or 150° (one-half of the thermometer-bulb dipping in the liquid), while at temperatures below 115° the bromiring was slow, and the amount of paraiodbenzylbromide formed small. These observations are important, because they tend to fix, in the case of paraiodtoluol, the limit at which bromine ceases to enter the main ring in any great amount, and thus one step is taken toward giving a more definite form to Beilstein's rule about the action of bromine on toluol and its homologues.* At temperatures above 150°, I observed the liberation of iodine, but, as already stated in the first paper on this subject, paraiodbenzylbromide was formed, although according to Beilstein † bromine enters the ring even at the boiling-point of toluol, if iodine has been added. This exception to Beilstein's rule has suggested a series of experiments, which will be carried on in this laboratory, to determine what effect, if any, the nature and position of the radical attached to the main ring exercise upon the bromiring of substituted toluols.

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* These Proceedings, Vol. XII. (n. s. IV.) p. 219.

The following substances were obtained from paraiodbenzylbromide by the usual metathetical reactions: —

Paraiodbenzylalcohol, $C_6H_4ICH_2OH$, was prepared from the product of the action of sodic acetate on the bromide, by heating it to 160° with aqueous ammonia in a sealed tube: the solid thus obtained was purified by pressing between filter-paper, and crystallization from carbonic disulphide. It was also made ‡ by boiling paraiodbenzylbromide with water for several hours in a flask with a return-condenser: that hydrobromic acid was set free, was proved by the acid reaction of the water, and the precipitate of argentic bromide formed on adding argentic nitrate to it, after the organic matter had been removed.

The alcohol prepared according to the first method was analyzed.

0.2508 gr. substance gave 0.3260 gr. CO_2 and 0.0633 gr. H_2O.

	Calculated for C_7H_6IOH.	Found.
Carbon	35.89	35.45
Hydrogen	2.99	2.79

Properties. It crystallizes from carbonic disulphide or alcohol in small silky white scales; from boiling water, in long needles, with an aromatic odor like that of the corresponding bromine compound, but no action on the eyes; melting-point, 71¾°; very slightly soluble in cold, somewhat more soluble in hot water; freely in alcohol, ether, benzole, and carbonic disulphide.

The *paraiodbenzylacetate* was not obtained pure, because, on account of its instability, a great quantity of substance would have been consumed in its preparation; and it did not seem worth while to sacrifice the large amount of time necessary to prepare so much paraiodbenzylbromide, especially as its decompositions seemed analogous to those of the more accessible parabrombenzylacetate, which will be studied in this laboratory. The following account of the attempts to prepare it is, however, given, as it throws some light upon the nature of the substance. The first difficulty encountered was due to the fact that the sodic acetate only partially decomposed paraiodbenzylbromide. The two substances were boiled with absolute alcohol, and an oily product obtained, which solidified at 0°, and after crystallization from alcohol melted at 45°: the results of an analysis of this substance are given under I.; it was then heated once more with sodic acetate and absolute alcohol, and the result analyzed, when the numbers given under II. were obtained.

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* These numbers have been tested and confirmed in this laboratory by Mr. A. W. Field, who will publish his results in detail, as well as similar experiments upon other substituted toluols, in a later paper of this series. — [C. L. J.]

† Ann. Chem. Pharm. 143, p. 369.

‡ These Proceedings, Vol. XII. (N. S. IV.) p. 221.

I. 0.4007 grs. substance gave 0.4416 gr. CO_2 and 0.0909 gr. H_2O.

II. 0.2936 gr. substance gave 0.3403 gr. CO_2 and 0.0738 gr. H_2O,

	Calculated $C_7H_6IC_2H_3O_2$.	Found. I.	Found. II.	Calculated C_7H_6IBr.
Carbon	39.12	30.05	31.61	28.28
Hydrogen	3.26	2.52	2.79	2.03

These numbers seem to show that the substance was nothing more than a mixture of the acetate and bromide, and this conclusion is strengthened by the fact that it attacked the eyes like the bromide. In order to obtain the acetate from this mixture, it would have been necessary to use fractional distillation, which had been found, in the case of parabrombenzylacetate, to cause almost complete decomposition: it was therefore thought better to try to decompose the paraiodbenzylbromide completely by using argentic, instead of sodic, acetate. In this case, the product was an oil, which showed no signs of solidification, even after standing in an open watch-glass for some time. The term then came to an end, and it remained exposed to the air during the summer vacation of three months, at the end of which time the watch-glass contained a solid residue, proved to be paraiodbenzoic acid by its melting-point and the following analysis of its silver salt:—

0.2185 gr. substance gave by precipitation with hydrochloric acid 0.0857 gr. AgCl.

	Calculated for $C_6H_4ICO_2Ag$.	Found.
Silver	30.42	29.62

The formation of this acid under these conditions can be explained by the supposition that the acetate absorbed water from the atmosphere, and became converted into the alcohol, which was then oxidized by exposure to the action of the air; and this view is supported by the observation, that a product from the reaction of paraiodbenzylbromide on sodic acetate dissolved in ordinary alcohol, when precipitated by water, and the oil thus obtained allowed to stand exposed to the air in a watch-glass, deposited crystals, which upon recrystallization melted at 70°, and were, therefore, the alcohol. It must, however, be remembered that this oil contained some of the bromide, and that the alcohol may have been formed from this instead of the acetate. This conversion of the acetate into the alcohol or acid seemed likely to render its purification so difficult that the experiment with argentic acetate was not repeated. A similar formation of the substituted benzoic acid has been observed in the attempts to purify parabrombenzylacetate.

Paraiodbenzylcyanide, $C_6H_4ICH_2CN$, was obtained by boiling the bromide with alcoholic potassic cyanide, precipitating with water, and crystallizing from alcohol. Its composition was established by the following nitrogen determination: —

0.3523 gr. substance gave 17.83 cc. nitrogen, under a pressure of 736. mm. and a temperature of 23°.

	Calculated for C_7H_6ICN.	Found.
Nitrogen	5.76	5.67

Properties. White plates with a pearly lustre, characteristic odor, and no action on the eyes; melting-point, 50½°; insoluble in water; readily soluble in alcohol, ether, benzole, carbonic disulphide, and glacial acetic acid.

Paraiodalphatoluylic acid, $C_6H_4ICH_2COOH$, was obtained by heating the cyanide with fuming hydrochloric acid to 100° in a sealed tube: after four hours, the liquid on cooling became filled with flattened needles, which were purified by crystallization from boiling water. The composition of the acid was fixed by the analysis of its silver salt.

Properties. Narrow, tapering, white plates, often over two centimeters long, with an agreeable odor resembling that of sweet alyssum; melting-point, 135°, sublimes in branching needles at a somewhat higher temperature; but slightly soluble in cold, readily in hot water, and in alcohol, ether, benzole, carbonic disulphide, and glacial acetic acid. A mixture of potassic dichromate and dilute sulphuric acid oxidizes it, forming paraiodbenzoic acid, which was recognized by its high melting-point. It is freely soluble in aqueous ammonia, but the ammonium salt could not be obtained in the solid state, as evaporation of the solution, whether on the water-bath or *in vacuo*, at ordinary temperatures, decomposed it, leaving the acid, which was recognized by its sparing solubility in cold water and its melting-point of 135°.

Argentic paraiodalphatoluylate, $C_6H_4ICH_2COOAg$, was made by adding the solution of the acid in ammonia water, from which the excess of ammonia had been driven off by warming on the water-bath, to argentic nitrate: the white curdy precipitate was washed, dried *in vacuo*, and analyzed.

0.7065 gr. substance, dissolved in dilute nitric acid and precipitated with hydrochloric acid, gave 0.2701 gr. of argentic chloride.

0.1214 gr. substance gave 0.0475 gr. of argentic chloride.

	Calculated for $C_8H_6IO_2Ag$.	Found. I.	Found. II.
Silver	29.27	28.78	29.45

Properties. A white curdy precipitate resembling argentic chloride, sparingly soluble in boiling water, from which it crystallizes in lustrous plates; readily soluble in dilute nitric acid. The dry salt is only slightly blackened by heat and light.

Baric paraiodalphatoluylate, $Ba(C_6H_4ICH_2COO)_2 . H_2O$, prepared by neutralizing baric hydrate with a hot aqueous solution of the acid, and evaporating to crystallization, gave the following results on analysis: —

0.5400 gr. substance dried *in vacuo* lost, when heated to 100°, 0.0153 gr., and gave 0.1881 gr. $BaSO_4$.

	Calculated $Ba(C_8H_6IO_2)_2 . H_2O$.	Found.
Water	2.66	2.83
Barium	20.23	20.48

It forms microscopic white needles, easily soluble in water.

The *calcium salt*, made by treating calcic carbonate with a concentrated aqueous solution of the acid, was also freely soluble in water, and crystallized in slender branching needles.

The solution of the ammonium salt of the acid obtained by dissolving it in aqueous ammonia, and driving off the excess of ammonia on the water-bath, behaved as follows with various reagents: —

With *cupric sulphate*, pale bluish-green flocks, insoluble in water, but soluble in acids.

With *ferric chloride*, pale yellowish-brown flocks, insoluble in water.

With *chromic chloride* or *nickelous nitrate*, a green precipitate.

With *cobaltous nitrate*, a pink precipitate.

With *plumbic acetate*, *mercurous nitrate*, or *aluminic chloride*, a white flocculent precipitate.

With *zincic* and *manganous nitrate*, salts which are sparingly soluble in water and crystalline; the manganese salt is flesh-colored. No precipitate was obtained with salts of *cadmium*, *magnesium*, *strontium*, and the *alkaline metals*. In general, its salts resembled very closely those of the corresponding parabromalphatoluylic acid.

Paraiodbenzylsulphocyanate, $C_6H_4ICH_2SCN$, was made by boiling an alcoholic solution of potassic sulphocyanate with paraiodbenzylbromide: the oil obtained by precipitating the product with water solidified upon cooling, and was purified by draining on filter-paper, and crystallization from hot alcohol.

0.3661 gr. substance gave 0.4702 gr. CO_2: the water was lost by breaking the chloride of calcium tube.

0.2159 gr. substance gave, according to Carius, 0.1844 gr. baric sulphate.

	Calculated for C_7H_6ISCN.	Found.
Carbon	34.91	35.02
Sulphur	11.64	11.72

Properties. It crystallizes from alcohol in long white lustrous plates which are often twinned; it has a slight but disagreeable odor, like that of the benzylsulphocyanate; melting-point, 40¼°; insoluble in water; slightly soluble in cold, more so in hot alcohol, freely in ether, benzole, carbonic disulphide, and glacial acetic acid.

PARAIODBENZYLAMINES.

Alcoholic ammonia removed the bromine from the paraiodbenzylbromide with great ease: in fact, it was only necessary to warm the substances together in a flask to obtain a voluminous white precipitate consisting of the secondary and tertiary amines. If a more dilute solution was used, the tertiary amine alone was deposited; while a mixture of this and the secondary amine fell upon adding water. The liquid decanted from this precipitate was evaporated to dryness, and the residue separated by treatment with water into an insoluble bromide of the secondary and a soluble salt, probably the bromide of the primary amine, which, however, could not be obtained in quantity sufficient for examination.

It would seem that the primary amine can be formed in larger quantities under certain conditions, as, on one occasion, a basic oil which solidified after some time was obtained from the wash-waters, but this was unfortunately lost before it was examined, and all attempts to prepare a fresh portion were unsuccessful. The mixture of the secondary and tertiary amines, after thorough washing with water, was treated with hot alcohol, which separated without difficulty the more soluble secondary from the but slightly soluble tertiary amine.

Triparaiodbenzylamine, $(C_6H_4ICH_2)_3N$, was purified by recrystallization from ether, dried *in vacuo*, and analyzed.

I. 0.3609 gr. substance gave 8.08 cc. nitrogen, under a pressure of 745.6 mm. and a temperature of 20°.

II. 0.8461 gr. substance gave 21.08 cc. nitrogen, under 742.8 mm and 21°.

	Calculated for $(C_7H_6I)_3N$.	Found. I.	II.
Nitrogen	2.15	2.51	2.76

Properties. White needles arranged in groups resembling hour-glasses, with an agreeable odor; melting-point, 114½°; is turned gray by heating, and the melting-point is then much higher; insoluble in water and cold alcohol; very slightly soluble in boiling alcohol, easily in ether, benzole, and carbonic disulphide. A chloride could not be obtained by treating a solution of the base with hydrochloric acid.

Triparaiodbenzylamine chlorplatinate, $[(C_6H_4ICH_2)_3NH]_2PtCl_6$, appeared as a bulky yellow precipitate, on adding a solution of platinic chloride to the amine dissolved in ether. It was washed with alcohol, dried *in vacuo*, and analyzed:—

0.7812 gr. substance gave on ignition 0.0886 gr. platinum.

	Calculated for $[(C_7H_6I)_3NH]_2PtCl_6$.	Found.
Platinum	11.32	11.34

Yellow needles nearly insoluble in water and alcohol.

Diparaiodbenzylamine, $(C_6H_4ICH_2)_2NH$, was purified by repeated recrystallization from hot alcohol, dried *in vacuo*, and analyzed.

0.4050 gr. substance gave 12.33 cc. nitrogen, under a pressure of 763.8 mm. and a temperature of 25°.5.

	Calculated for $(C_7H_6I)_2NH$.	Found.
Nitrogen	3.12	3.40

Properties. White needles with square ends, having an odor somewhat resembling that of the nitrile; melting-point, 76°; insoluble in water; sparingly soluble in cold, freely in hot alcohol, and in ether, benzole, and disulphide of carbon.

The *chloride of the base*, obtained by adding hydrochloric acid to its alcoholic solution, crystallizes in thick, white plates, with a very high melting-point; nearly insoluble in water, slightly soluble in alcohol and benzole, freely in carbonic disulphide and glacial acetic acid. It was proved to be the chloride, by treating its nitric acid solution with argentic nitrate, when argentic chloride was precipitated.

The *bromide of the base* was formed under certain conditions during the preparation of the amines: it was purified by crystallization from alcohol. Short, thick, pointed, white prisms, with a pearly lustre, and a high melting-point; insoluble in water; somewhat soluble in alcohol, more so in ether, benzole, and carbonic disulphide; sparingly soluble in glacial acetic acid. It was proved to be the bromide of the diamine, by treating it with a solution of sodic hydrate, when a base was set free, melting after recrystallization from boiling alcohol at 76°; while bromine was detected in the sodic hydrate by the usual test with chlorine water and carbonic disulphide.

Diparaiodbenzylamine chlorplatinate, $[(C_6H_4ICH_2)_2NH_2]_2PtCl_6$, was obtained as a yellow precipitate on adding platinic chloride to an alcoholic solution of the base: it was washed with alcohol, dried *in vacuo*, and analyzed:—

0.3951 gr. substance gave on ignition 0.0609 gr. platinum.

	Calculated for $[(C_7H_6I)_2NH_2]_2PtCl_6$.	Found
Platinum	15.07	15.42

Properties. Pale yellow microscopic crystals, grouped like certain forms of frost, almost insoluble in water and alcohol.

XVI.

REMARKS ON THE BRAIN,

ILLUSTRATED BY THE DESCRIPTION OF THE BRAIN OF A DISTINGUISHED MAN.

BY THOMAS DWIGHT, M.D.,

Late Professor of Anatomy at the Medical School of Maine.

Read Dec. 12, 1877.

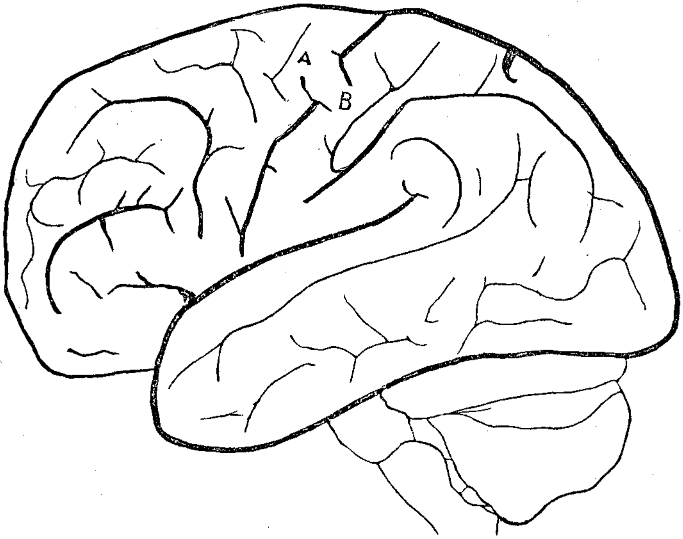
THE objects of this paper are: first, to describe the brain of a distinguished man, for in the present state of knowledge, when we are ignorant to what extent purely anatomical appearance may be of psychological or physiological significance, the observation of the brains of known individuals is doubly important; secondly, to call attention to an extremely rare anomaly of the convolutions; and, lastly, to present a few observations on the extent of our knowledge of the brain.

The late Mr. Chauncey Wright, whose brain is the one to be described, died in the prime of life. He was a man of very varied acquirements, a proficient in physics and mathematics, and was what may be called a general critic. He was considered an instance of very exceptional mental power. He was of rather large frame, with a large head and a high forehead.

The brain weighed 53½ oz. avds. The most striking point in the shape is the height in the frontal region and the sharpness of the curve where the upper surface passes into the anterior one. In most brains the two ends are in this respect nearly symmetrical, but in this one the difference is very marked. The convolutions are large and plump, with deep fissures between them; but the small, irregular fissures, that give many brains a very complicated appearance, are comparatively few except in the frontal region. The two sides are as symmetrical as are often observed, the chief difference between them being the somewhat greater complexity of the left frontal lobes.

The frontal convolutions are the most complex. On each side, the first one arises by two roots from the anterior central convolution.

The second is crowded outward, and arises in common with the third. The first is a good deal cut up by secondary fissures. The under surface of the frontal lobe is very simple, especially on the right side. In the parietal region, the superior parietal lobe (of Ecker, the præcuneus of Bischoff), is perhaps uncommonly large. On the left side, it sends a narrow prolongation far down behind the posterior central convolution. The arrangement of the convolutions turning round the fissure of Silvius and running to the apex of the temporal lobe is remarkably simple, though, according to Bischoff, this part is usually complicated in European (*i.e.*, Caucasian) brains: the one in question is in this respect even simpler than that of the "Hottentot Venus." There is nothing important to record concerning the occipital lobe. The median surface may be briefly discussed. The right *fissura calloso-marginalis* is interrupted by a bridge on the upper part of its course. The occurrence of a bridge is not uncommon, but usually it is placed lower down in front of the *corpus callosum*. This fissure, after turning up behind the posterior central convolution, runs a considerable distance into the præcuneus, farther on the right than on the left.



This drawing, though made from the brain, is meant rather as a diagram than as an accurate representation. The letters A and B are placed respectively on the anterior and posterior central convolutions. The anomalous bridge is between them.

Few of these points are of much consequence. We might dismiss the brain with the statement that the frontal region is largely developed and complicated and the rest simple, were it not for the very rare anomaly about to be described. This is a small gyrus uniting the two central convolutions by dividing the fissure of Rolando. It occurs on both sides of the brain. On the left, it is situated about one inch from the median fissure, and runs obliquely forward and upward from the posterior to the anterior central convolution. It is superficial throughout and absolutely unmistakable. On the right, it is much less easily recognized: for, though superficial, it is very near to the median fissure, and at first suggests simply a somewhat premature ending of the fissure of Rolando; but a glance at the inner side of the hemisphere shows that this view is not tenable, — that there is actually a bridge, and that the fissure is even a little longer than usual. The termination of the calloso-marginal fissure is a useful guide in studying these relations. When the writer examined this brain, there was but one case of this anomaly on record; and this, curiously enough, was in the brain of a known man, that of Dr. Fuchs, of Göttingen, which is described by Wagner.* "Both [*i.e.*, the central convolutions] are connected with one another by bridges, of which, especially on the left side, a very considerable one arises, with a broad root, from the anterior central convolution." It will be seen by consulting Wagner's plates that the gyrus on the left side of Fuchs's brain is very similar to the one described in this paper. Bischoff † refers to Wagner's statement in a tone approaching that of unbelief. He writes: "These two convolutions [the central ones] have always two communications around the ends of the fissure, bordering on the great median fissure and on the Sylvian fossa, but *never* in their course. It is very striking that R. Wagner should describe and figure such a communication between both central convolutions on the brain of Professor Fuchs, as if it were something of frequent occurrence. In the many brains that I have examined, I never saw any thing of the kind." And, again, Bischoff, speaking in another place of the same fissure, says, "Which [the fissure] is distinguished from all other fissures by its early appearance, its unchanged direction and structure, and the fact it is never interrupted by any convolution, and only very gradually inclines rather more backward." And again, in discussing this fissure in apes, "Its course in apes also is never interrupted." Ecker[*] states that the fissure "is never or extremely seldom bridged over in its course by a secondary convolution," and in a foot-note mentions that such an occurrence has never been observed by Turner or Bischoff. The next to report similar cases is Féré,[†] who has seen two; in one of them, however, the bridge is situated much lower, and, for reasons to be given later, should perhaps be excluded from this class. We give his brief account in his own words: "Le sillon de Rolando peut être interrompu aussi par des plis de passage. Nous avons vu deux cerveaux sur lesquels les deux circonvolutions ascendantes étaient réunis par un pli de passage aussi saillant qu'elles et absolument continu. Sur l'un ce pli était situé à l'union du tiers inférieur avec les deux tiers supérieurs du sillon de Rolando. Sur l'autre il était situé vers la partie moyenne, de sorte qu'il formait avec les deux circonvolutions ascendantes une *H* inclinée en arrière. (Ces deux sujets n'avaient présenté aucun trouble intellectuel.)" Very recently, Heschl,[‡] of Vienna, comes on the field with a preliminary paper, announcing some of the results of the examination of 1,087 brains, 632 of which were from male bodies, and 455 from female. In these he has seen the anomaly six times: three times on the right and twice on the left in male brains, and once on the left in a female one. Heschl has the merit of being the first to explain the occurrence of this phenomenon. With the exception of one of Féré's cases, the bridge was always near the upper end of the fissure of Rolando; and Heschl has observed that at about the junction of the middle and upper thirds of the fissure there is very frequently a transverse gyrus in its depths, which is not visible till the central convolutions are pulled apart. He has found this in his 1,087 brains, when it reached only from one-sixth to one-third of the way to the surface, 75 times; when it reached from one-third to five-sixths of that distance, 67 times; and, as already stated, six times when it was on a level with the surface. Since reading Heschl's paper, the writer has examined a number of brains, and has found several instances of a deep gyrus in this situation. It seems difficult to deny the force of Heschl's argument, that these rare anomalies are instances of uncommon development of this fold.

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* Vorstudien zu einer wissenschaftlichen Morphologie und Physiologie des menschlichen Gehirns als Seelenorgan. 1860–62.

† Die Grosshirnwindungen des Menschen. 1868.

This, then, is an anatomical fact of considerable curiosity, that de-serves to be recorded; but the question presents itself, What is its significance and importance? which introduces necessarily the larger question, What is the significance of the convolutions? It has long been believed that the weight of the brain, and the complexity of the convolutions, are in direct ratio to the intellectual power of the individual; but of late statistics have gone far to overthrow the former of these doctrines, and to weaken belief in the latter. The series of weights of nearly a thousand brains tabulated by Wagner, and the list of weights of well-known men given by Flint, seem to show that weight is of but little importance; and the theory of the convolutions rests chiefly on the fact that the brains of idiots are but slightly developed. Certain is it that we have not the data to establish the theory.

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[*] The Cerebral Convolutions of Man. 1869.

[†] Note sur Quelques Points de la Topographie du Cerveau, par Ch. Féré. Archives de Physiologie Normale et Pathologique. Paris, 1876.

[‡] Wiener Medizinische Wochenschrift, Oct. 13, 1877.

A difficulty, however, which has, we think, been very much overlooked, but which nevertheless lies at the root of the whole matter, is that we are dealing with words that convey no definite idea. We say that a heavy brain accompanies intellect, intelligence, a gifted mind, but have merely the vaguest idea what we mean by it. Almost if not quite all truly distinguished men are noted for their ability in some special direction, often counterbalanced by marked weaknesses in others. The ability of the mathematician, the musical composer, the novelist, the politician, the actor, the strategist is in each case different, and we are not certain in which it is the highest. We are also ignorant, in spite of the labor expended on the subject, how much the size of the brain depends on that of the body, and whether active muscular exercise, which enlarges the muscles, may not, *pari passu*, enlarge the central nervous organs. Another point to be considered is the effect of opportunity, not only in making merit known, but, what is far more important, in developing it. This question, indeed, is of primary importance: for if it be true that the brain has very nearly reached its anatomical perfection at the age of eight years, and increases but slightly up to twenty, and but very slightly subsequently (Marshall), and if it be true that its shape or size is any index of the mind, it must be an index of the mind in the rough, or, to speak more accurately, of its possibilities; for it certainly has not gained its full strength at twenty. Bad habits or want of education may not only prevent an originally good mind from doing itself justice, but may make it incapable of even ordinarily good work; yet there is no reason to suppose that the weight or outline of the brain would be modified.

Mr. W.'s brain can hardly be quoted in support of existing theories. If the general estimate of his abilities be just, — as we believe it is, —

and if weight of brain be any criterion of mental power, we certainly should expect one of greater weight than 53½ oz. In the same way, we are surprised to find the great simplicity of a part (the parieto-temporal region) which we are told is usually complicated in European races. According to Wagner, a complicated — which with him is synonymous with a highly-developed — brain is of the same nature throughout; but here is a marked exception to the rule, if rule it be. Comparing this brain to some of known men figured by Wagner, we have little hesitation in declaring it decidedly more simple than those of Gauss and Derichlet, mathematicians, and Fuchs, a physician; rather more simple than that of Hermann, a philologist, and much in the style of that of a celebrated naturalist, whose name is not given.

It must, of course, be admitted that a certain amount of cerebral matter is necessary to make a man more than an idiot; but, this being granted, we think that in consequence of our uncertainty of what mental elements constitute what is vaguely called a great mind; in consequence of our ignorance of many qualities of any given mind, of the opportunities of any given individual, and the various influences which must obscure our knowledge of his real character; in consequence of the apparently contradictory results of statistics of the weight of brains, and of our ignorance of how much that depends on the weight of the body; in consequence, finally, of the unsatisfactory results of the examination of the convolutions, — we must admit that as yet we have no proof of any definite relation between the weight and shape of the brain on the one hand and the mental capacity on the other.

XVII.

THEORY OF ABSORPTION-BANDS IN THE SPECTRUM, AND ITS BEARING IN PHOTOGRAPHY AND CHEMISTRY.

BY DR. ROBERT AMORY.

Presented Jan. 9, 1878.

IN order to present this communication clearly, I must apologize for reminding you of certain facts, which probably are familiar to you as well as to myself; and these may be summarized in the following review. The rays of light absorbed by a colored solution cannot always be determined by its apparent color; for instance, an aniline alkali blue salt of commerce will extinguish or neutralize green and yellow rays (between *b* and *D* lines in the solar spectrum), eosin (fluorescein) of commerce absorbs only the green rays, its greatest intensity being at the *E* lines, it should also be observed that this latter substance by transmitted light appears red, and by reflected light fluorescent green.

By the term "absorption" it is ordinarily meant that a colored solution by transmitted light allows only those light rays to pass which do not belong to itself: in other words, certain of these rays are extinguished or neutralized, whilst others are transmitted. To determine correctly which of these rays are absorbed and which are transmitted, we must examine, by means of a glass or other transparent prism, the sunlight (or other incandescent light) transmitted through the colored solution.

The explanation of absorption is usually given somewhat in the following manner: —

A colored solution owes its absorption-bands, seen in the spectrum from its transmitted sunlight, to the fact that the ether molecules are excited by the sunlight, or other source of illumination, to move in undulatory vibrations; and that certain of these waves are of the same length with those of the solution, and hence are extinguished or withheld, whilst all others of unequal length are allowed to pass through. This explanation is founded upon the analogy offered by the solar or Frauenhofer lines, which appear black because their monochromatic rays pass through the gaseous vapor of the same substances placed between our eyes and the point of their combustion; and thus the original monochromatic rays are absorbed by this gaseous vapor.

The theory of absorption of colored solutions may also be explained, by assuming that certain of the rays excited in the solution are the result of the motion of waves of unequal length. Now, if we call to mind that each ray gives its own illumination or image of the slit, and consequently that there are side by side a series of these images, whose illumination is the result of the ether molecules vibrating in wave-lengths gradually increasing in length, — for instance, at the solar line H' being 3,928 according to Ångstrom's map, or $\frac{174}{10,000,000}$ in. to 7605 or $\frac{266}{10,000,000}$ in. at A line, — we may easily conceive that certain of these ether molecules, passing through a solution whose molecules are of a size capable of receiving their motion, transfer their energy; whilst certain others of different length, both of longer and shorter size, only partially transfer their energy; and again certain others, not losing any of their energy, because they do not excite any motion of their own kind, pass through at the same initial rate. The consequence would follow, that the first set of waves become extinguished, and hence their illumination ceases, after their projection into the solution; the second rays are partially quenched; and the third set appear as bright as when first projected. Thus, instead of seeing the absorption-bands of the spectrum transmitted through a colored solution, always with definite limits, in some instances we observe that the greatest amount of light absorbed is at the middle, and shades lighter towards the outside limits of these bands. The energy consumed in extinguishing certain rays of light must, then, by the law of conservation of energy, become stored up in a latent form in all those colored substances which give absorption-bands.

Herrmann Vogel (Jahresbericht, 1861, and in Watt's Dict. of Chem. vol. v. p. 295) maintained that silver iodide or bromide or chloride are reduced by certain rays of solar light to the subiodide, subbromide, and subchloride; or, in other words, —

$$2\ (\text{Ag. Br.}),$$

by exposure to those refrangible rays of light which to our eye appear violet or blue, becomes

$$\text{Ag.}^2\ \text{Br.},$$

and that one portion of bromine is set free.

A year or two ago, Vogel, as has already been referred to by me, in a previous communication to this Academy, observed that the addition of certain pigments, chiefly aniline, to the above-named silver bromide salt, would cause a partial reduction of the silver salt, when these were exposed to rays of less refrangibility than the blue. He explained this phenomenon by supposing that the pigment corallin or fuchsin, which has a red color, absorbed and stored up latent energy, and so re-enforced the primary action on the silver bromide. Captain Abney, as has also been stated, informed me last year that the addition of gum benzoin increased likewise the sensitive action on the silver bromide, so that by its addition he could photograph the less refrangible rays of the solar spectrum. He has publicly, in his South Kensington address, explained this increased action on the principle that the molecules of silver bromide rotating and vibrating at a given rate were weighted down and moved at a slower rate, so that the interference of rays of less refrangibility than what constitutes blue light would reduce the strained bromide silver salt, and so give an image from green and yellow light. This he explained on the principle that these silver bromide molecules consequently could be made to swing in discord with waves of greater amplitude.

If Vogel's explanation is correct, we should suppose that any pigment which absorbs the green or yellow rays, and does not prevent the chemical process of reduction, would likewise increase the sensitiveness of the silver salt to these rays. Now, the same aniline blue that I mentioned at the beginning of this communication absorbs green and yellow rays, and does not prevent the reduction of silver bromide to rays of blue light. This blue-stained silver bromide was exposed to the solar spectrum from about line *G* to line *A*, and yet I could obtain no image below line *F* in the blue. Again, if a silver bromide emulsion (so called) be stained with the same aniline colors which Vogel himself used, — viz., fuchsin or corallin, — we ought to have the silver salt reduced on exposure to the green rays of light. Unless there be free nitrate of silver in the emulsion, we get no such effect. These two experiments conflict with Vogel's explanation. Captain Abney states that, if there is an excess of silver bromide, the addition of the aniline does not increase its sensitiveness to the less refrangible rays. Now, if there is an excess of silver bromide, there can be no free silver nitrate; and, unless there be free nitrate, there is no action from the pigment and silver bromide. The explanation of this photographical action of the green and yellow rays of light must be sought out in some other way.

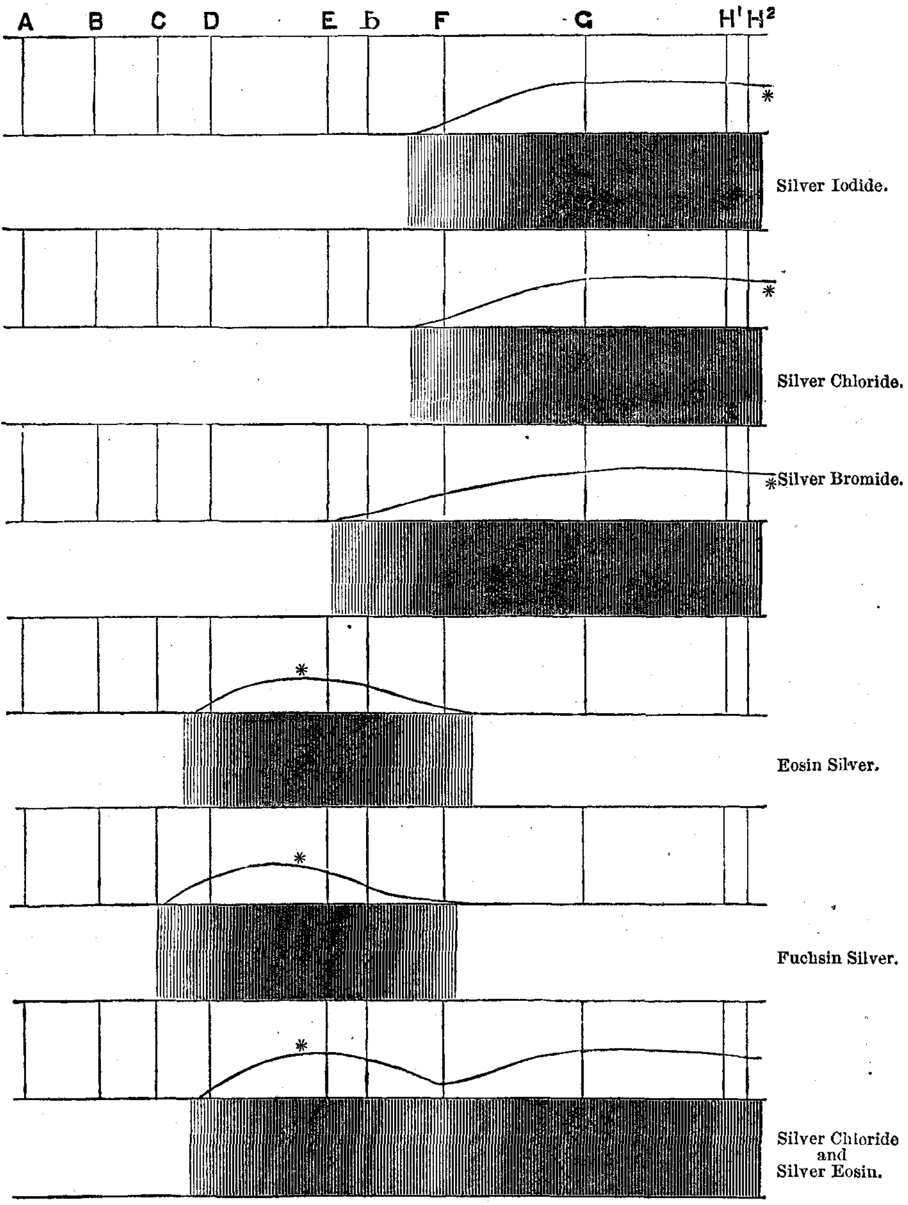
If an aqueous solution of eosin (fluorescein, a rather complicated organic compound) be added to a neutral solution of silver nitrate, a colored precipitate is thrown down. It seems to me that the precipitate obtained from the addition of silver nitrate to the eosin is a definite salt of silver; and this is shown in the following way: Wash the precipitate of eosin and silver nitrate with distilled water, until all trace of an excess of silver nitrate is removed; then dissolve this washed precipitate in a strong solution of sodium hyposulphite, until the latter is saturated; then add to this saturated solution a solution of cadmium or potassium bromide. A precipitate is thrown down, which is again soluble in more of the sodium hyposulphite. Now, if this salt of silver eosin be precipitated upon a film which contains neither bromide, chloride, or iodide of silver, a definite picture of the green rays of the spectrum will be obtained. This image which I now exhibit corresponds to that part of the spectrum which this silver salt absorbs, as may be seen on comparing it with the absorption spectrum of an emulsion of this salt which has been prepared for that purpose.

Commercial aniline chloride is a salt of which the coloring matter in solution absorbs violet, and also the green rays between *F* and *D* lines of the solar spectrum; its greatest intensity being at *E* line of the solar spectrum. Here is a definite image of those rays, which corresponds with absorption-bands of this silver salt. All three plates were exposed to the spectrum for about ten minutes only. We will go still further. Silver iodide absorbs the violet and the more refrangible blue rays. It is these rays only that reduce this silver salt. In other words, 2 (Ag. I.) is reduced to Ag^2 I., on its exposure to the rays of light which this salt absorbs. Argentic chloride, on exposure to those refrangible blue rays which it absorbs, becomes argentous chloride. Argentic bromide, on exposure to those refrangible blue rays (down to line *F*) which it absorbs, becomes argentous bromide. The intensity of photographical action corresponds to that of absorption. Neither of these salts are affected by rays of less refrangibility than those which they absorb; yet there may be slight action extending below the invisible band, corresponding with an extended and faint absorption below this point. I would therefore deduce a general proposition founded on these experiments, and expressed in the following terms: —

A colored silver salt is reduced by rays of light of the same refrangibility which it absorbs.

I would propose the following theory as being very probable: The colored silver salt owes its cohesion to the fact that the combined salt. when exposed to light, has a molecular vibration, expressed in waves of definite length. The addition of rays of light which may be in discord or in accord with the vibration disturbs the cohesion, and hence either the whole or part of the combining acid is set free. The annexed diagram will serve to illustrate the relation between the absorption and photographical action of some of the silver salts.

By using a long focus collimating lens, practical experience shows that not only do we obtain more illumination of the spectrum, but that we can also bring into vision the ultra violet and red rays; so that the solar lines to *L* can be distinctly seen, as also the *A* lines.



* Lines showing curve of intensity of absorption and photographical action.

XVIII.

SURFACES OF THE SECOND ORDER, AS TREATED BY QUATERNIONS.

THE THESIS OF A CANDIDATE FOR MATHEMATICAL HONORS CONFERRED WITH THE DEGREE OF A.B., AT HARVARD COLLEGE, AT COMMENCEMENT, 1877.

BY ABBOTT LAWRENCE LOWELL.

Presented by Professor Benjamin Peirce, Jan. 9, 1878.

THE surfaces of the second order, or *Quadrics*, as they are very commonly called, present by far too great a field to be investigated in every part in any single thesis. I have therefore chosen only a few branches of the subject; and I have been guided in the selection chiefly by a desire to avoid, as much as possible, those portions of the subject which have been the most thoroughly treated by Hamilton. With this object in view, I have passed over entirely the vast field of *foci* and *confocal surfaces*, and have touched but slightly upon *cyclic normals* and *asymptotic cones*. I have been especially attracted to consider the relations existing between the various conjugate lines and planes of any quadric, and the general relations which the different classes of quadrics bear to each other.

It has also been my object to exhibit that variety of expression which is so peculiar to quaternions, by approaching all questions from more than one point of view. With this idea, I have studied many of the cases arising under the self-conjugate function $\varphi\rho$ under both the *cyclic* and the *rectangular* forms, showing how these forms give different expressions to the same result. And finally, considering it a great advantage to be as general as possible in the treatment of any mathematical subject, I have tried to keep the self-conjugate function under the general form $\varphi\rho$, without attending to the special forms of the terms which compose it.

The equation, $$S\rho\varphi\rho = \text{constant},$$

where $\varphi\rho$ is any vector function of ρ, represents in general a surface; for if we write

$$\rho = xi + yj + zk,$$

and if we assume any arbitrary values for x and y, we shall have a scalar equation to determine the corresponding value of z. Our equation, then, represents a surface for the same reason that any one equation between Cartesian co-ordinates represents a surface. It is almost needless to add that, since $\varphi\varrho$ may be any vector function of ϱ, the converse proposition is true: that any surface may be expressed by an equation of the form

$$S\varrho\varphi\varrho = \text{constant.}$$

The degree of the surface is higher by unity than the degree of the function $\varphi\varrho$, — understanding, as usual, by the degree of a surface the greatest number of times it can be cut by a right line. For suppose that $\varphi\varrho$ is of the n^{th} degree, and we want the intersections of the surface with the line,

$$\varrho = x\alpha.$$

$\varphi\varrho$ may be divided into the sum of vector functions, each of which is homogeneous with regard to ϱ, the highest being of the n^{th} degree. We shall thus have

$$S\varrho\varphi\varrho = S\varrho\varphi'\varrho + S\varrho\varphi''\varrho + \&c.$$

$$= T\varrho T\varphi'\varrho \cos {<}^{\rho}_{\phi'\rho} + T\varrho T\varphi''\varrho \cos {<}^{\rho}_{\phi''\rho} + \&c. = c.$$

Now we may substitute $x\alpha$ for ϱ in this equation. But $\cos {<}^{\rho}_{\phi'\rho}$ depends only on the direction of ϱ and $\varphi'\varrho$, and since we may write

$$\varphi'\varrho = \varphi'(T\varrho U\varrho) = T^{n}\varrho\varphi'(U\varrho),$$

the direction of $\varphi'\varrho$ depends only on that of ϱ. Hence, the cosines in the above equation are independent of x.

Now $T\varphi\varrho$ contains x to the same degree in each term as it does ϱ; that is, to the n^{th} degree in the highest term. Thus the equation

$$T\varrho T\varphi'\varrho \cos {<}^{\rho}_{\phi'\rho} + T\varrho T\varphi''\varrho \cos {<}^{\rho}_{\phi''\rho} + \&c. = c$$

is an algebraic equation of the degree $(n + 1)$, which gives $(n + 1)$ solutions for x, or $(n + 1)$ distances at which the surface is cut by the line

$$\varrho = x\alpha.$$

A surface of the second order may then be represented by an equation of the form

$$S\rho\varphi\rho = c,$$

where φ is a vector function of ρ of the first order.

The most general form of such a function, $\varphi\rho$, will be a function homogeneous in ρ plus a constant vector γ. But the homogeneous function is equivalent to a self-conjugate function $\varphi_0\rho$, plus a term of the form $V\varepsilon\rho$ (Hamilton's Elements, § 349, (4); Tait, § 174). Now we see that

$$S\rho\varphi\rho = S\rho(\varphi_0\rho + V\varepsilon\rho + 2\gamma) = S\rho\varphi_0\rho + 2\ S\gamma\rho = c,$$

writing in the 2 merely for convenience. That is, the homogeneous part of the function may be taken as self-conjugate. If we can next transform the origin to such a point that γ disappears, the surface will be represented by the equation

$$S\rho\varphi_0\rho = c,$$

and in this case all the variable terms will contain ρ to the second degree, so that satisfied by $+\rho$ the equation will also be satisfied by $-\rho$; *i.e.*, the origin will be at the centre. To find this point, write $\rho + \delta$ for ρ, and (dropping the suffix of $\varphi_0\rho$, but remembering that the function is self-conjugate)

$$S\rho\varphi\rho + 2\ S\rho\varphi\delta + 2\ S\gamma\rho + S\delta\varphi\delta + 2\ S\gamma\delta = c.$$

The terms $2\ S\rho\varphi\delta$ and $2\ S\gamma\rho$ take the place of the old term $2\ S\gamma\rho$; and, in order that they may disappear, we must have

$$S\rho(\varphi\delta + \gamma) = 0;$$

or, since ρ may have any direction,

$$\varphi\delta + \gamma = 0.$$

This is the condition that must be satisfied in order to transform the origin to the centre; and it gives in general a single finite solution for δ. I shall consider later the cases when this solution is indeterminate or infinite, and the corresponding form of the surface.

Suppose now that the equation

$$\varphi\delta + \gamma = 0$$

has been solved, and the centre found. Our equation then assumes the form

$$S\rho\varphi\rho = -S\delta\varphi\delta - 2\,S\gamma\delta + c,$$

and we may write it $S\rho\varphi\rho = c.$

By this process we have destroyed the three arbitrary constants involved in γ, and left only the six belonging to the self-conjugate $\varphi\rho$ (Ham. Elem., § 358). This is precisely what should happen, for the general equation of the second degree in Cartesian co-ordinates contains nine arbitrary constants, while by taking the centre as origin three of them are lost.

If in the transformed equation

$$S\rho\varphi\rho = c,$$

the constant vanishes, the equation represents a cone, since we may give any value to the tensor of ρ, as the equation is homogeneous. This case also I shall consider later. If the constant term does not vanish, we can divide by it, and get our equation in the more convenient form

$$S\rho\varphi'\rho = 1,$$

c disappearing into the new self-conjugate function $\varphi'\rho$.

If we differentiate

$$S\rho\varphi\rho = 1,$$

we find (Tait, §§ 132, 251 *c*)

$$S\rho\varphi d\rho + Sd\rho\varphi\rho = 0;$$

and since $\varphi\rho$ is self-conjugate,

$$Sd\rho\varphi\rho = S\rho\varphi d\rho,$$

or $$Sd\rho\varphi\rho = 0.$$

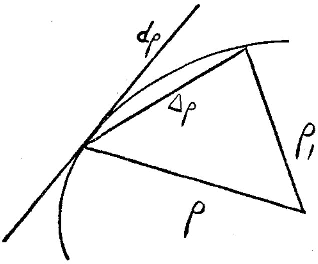


FIG. 1.

But $d\varrho$ is in the direction of the variation of ϱ at any instant. It is then in the direction of the tangent, at the extremity of ϱ (Fig. 1) (Tait, § 36).

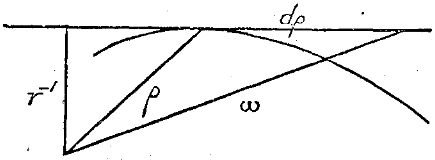


FIG. 2.

Now if we consider ϱ fixed, but allow $d\varrho$ to vary, we may write $(\omega - \varrho)$ for $d\varrho$ (Fig. 2), and the equation

$$Sd\varrho\varphi\varrho = S(\omega - \varrho)\varphi\varrho = 0$$

is that of a plane containing any tangent $d\varrho$; and, therefore, of the tangent plane. Since the extremity of ϱ is on the surface, we have

$$S\varrho\varphi\varrho = 1,$$

and the equation of the tangent plane may be written

$$S\omega\varphi\varrho = 1.$$

We see (Tait, § 205) that $\varphi\varrho$ is perpendicular to this surface, or, in other words, in the direction of the normal; and if we take ω in the direction of $\varphi\varrho$, or $= x\varphi\varrho$,

$$Sx\varphi\varrho\varphi\varrho = x(\varphi\varrho)^2 = 1;$$

$$\therefore\ x\varphi\varrho = \frac{1}{\phi\rho} = -\frac{\phi\rho}{T^2\phi\rho};$$

and this last is the perpendicular from the centre on the tangent plane.

CONJUGATE DIAMETERS AND DIAMETRAL PLANES.

If we want to find a line ω through the origin which bisects all chords parallel to another line α, $(\omega + x\alpha)$ and $(\omega - x\alpha)$ must both terminate in the surface: that is, ω must satisfy the equation

$$S(\omega \pm x\alpha)\varphi(\omega \pm x\alpha) = 1,$$

where x is a scalar. Now, if we develop this equation, we find

$$S\omega\varphi\omega + x^2S\alpha\varphi\alpha \pm 2\,x\,S\omega\varphi\alpha = 1.$$

But this is evidently impossible, unless

$$S\omega\varphi\alpha = 0.$$

The locus of ω is, then, a plane perpendicular to $\varphi\alpha$ and passing through the origin. It is also parallel to

$$S\varrho\varphi\alpha = 1,$$

which is the tangent plane at the extremity of a vector through the origin parallel to α. Conversely, if

$$S\omega\varphi\alpha = 0,$$

the locus of ω is a plane bisecting all chords parallel to α, because some x, a scalar function of ω, can evidently be found, such that

$$S(\omega \pm x\alpha)\varphi(\omega \pm x\alpha) = 1.$$

Since φ is self-conjugate, we have

$$S\alpha\varphi\omega = S\omega\varphi\alpha = 0,$$

so that the relation is reciprocal; and if ω be constant, and α vary, the locus of the latter is a plane parallel to the tangent plane at the extremity of a diameter parallel to ω. If β is any vector lying in the first plane, our two planes will be denoted by

$$S\omega\varphi\alpha = 0$$

and

$$S\omega\varphi\beta = 0,$$

and we have

$$S\alpha\varphi\beta = S\beta\varphi\alpha = 0.$$

The intersection of the two planes is $V\varphi\alpha\varphi\beta$, because this satisfies both equations: for

$$S\varphi\alpha\varphi\beta\varphi\alpha = 0 = S\varphi\alpha\varphi\beta\varphi\beta.$$

And, denoting this intersection by γ, we see that

$$S\alpha\varphi\gamma = S\gamma\varphi\alpha = 0,$$

and

$$S\beta\varphi\gamma = S\gamma\varphi\beta = 0.$$

Now

$$S\omega\varphi\gamma = 0$$

is the equation of a plane through the origin perpendicular to $\varphi\gamma$, and bisecting all chords parallel to γ. But such a plane must be that of α and β, for

$$S(x\alpha + y\beta)\varphi\gamma = xS\alpha\varphi\gamma + yS\beta\varphi\gamma = 0.$$

Thus we have a system of three planes, each of which bisects all chords parallel to the intersection of the other two. Hence, if three vectors α, β, γ are such that

$$S\alpha\varphi\beta = 0 = S\beta\varphi\alpha,$$

$$S\alpha\varphi\gamma = 0 = S\gamma\varphi\alpha,$$

$$S\beta\varphi\gamma = 0 = S\gamma\varphi\beta,$$

the diameters parallel to α, β, γ are conjugate diameters, and the planes,

$$S\omega\varphi\alpha = 0,\ S\omega\varphi\beta = 0,\ S\omega\varphi\gamma = 0$$

are conjugate diametral planes. The above equations, which may be written in the form

$$\cos \angle {}^{\phi\alpha}_{\beta} = 0,\ \cos \angle {}^{\phi\gamma}_{\alpha} = 0, \text{ and } \cos \angle {}^{\phi\beta}_{\gamma} = 0,$$

give three conditions for determining the directions of α, β, γ; since the direction of $\varphi\varrho$ depends only on the direction of ϱ. But three directions involve six arbitrary constants, of which we see that three may be selected arbitrarily. Thus, if one diameter, or one plane, be chosen, the other two can still be taken in an infinity of ways.

Again γ, for instance, bisects all chords through it parallel to the plane of α and β; because, if $\delta = a\alpha + b\beta$,

$$S\gamma\varphi\delta = aS\gamma\varphi\alpha + bS\gamma\varphi\beta = 0.$$

Hence the equation

$$S(m\gamma \pm x\delta)\varphi(m\gamma \pm x\delta)$$

$$= m^2S\gamma\varphi\gamma + x^2S\delta\varphi\delta \pm 2\,mxS\gamma\varphi\delta$$

$$= m^2S\gamma\varphi\gamma + x^2S\delta\varphi\delta = 1,$$

is satisfied by equal and opposite values of x.

Principal Diameters.

For any self-conjugate function $\varphi\varrho$, there are three real directions at right angles to each other, and in general only three directions, for which $\varphi\varrho$ is parallel to ϱ (Ham. Elem., § 354). We have already seen that $\varphi\varrho$ has the direction of the normal at the extremity of ϱ. If, then, ϱ is in any one of the three rectangular directions for which $\varphi\varrho$ is parallel to ϱ, its tangent plane must be parallel to the plane of the other two; which must therefore bisect all chords parallel to ϱ. These three directions are, therefore, those of a set of conjugate diameters.

We can see the same thing in a purely analytical way. Let i, j, k represent unit-vectors in the three rectangular directions determined by the above condition; and let $\varphi i = -c_1 i$, $\varphi j = -c_2 j$, $\varphi k = -c_3 k$. Then

$$Si\varphi j = -c_2 Sij = 0,$$

$$Sj\varphi k = -c_3 Sjk = 0,$$

$$Sk\varphi i = -c_1 Ski = 0.$$

Conversely, if α, β, γ are mutually rectangular, they must be respectively parallel to $\varphi\alpha$, $\varphi\beta$, $\varphi\gamma$. There is thus one set, and in general only one set, of conjugate diameters which are mutually rectangular.

The reciprocals of the scalar coefficients c_1, c_2, c_3, are the squares of the semiaxes of the quadric; for

$$S\varrho\varphi\varrho = T^2\varrho S\alpha\varphi\alpha,$$

where α is an unit vector in the direction of ϱ. But in the direction of a principal diameter, as i: —

$$T^2\varrho S\alpha\varphi\alpha = -T^2\varrho Sic_1 i = c_1 T^2\varrho = 1.$$

Hence,
$$T\varrho = \frac{1}{\sqrt{c_1}},$$

and the semiaxis is $c_1^{-\frac{1}{2}}\, i$. In the same way, the other semiaxes are $c_2^{-\frac{1}{2}} j$ and $c_3^{-\frac{1}{2}} k$.

If one of the c's is negative, its square root is imaginary, and therefore the radius vector does not cut the surface in that direction, and the quadric is a *single sheeted hyperboloid.* If two c's become negative, only one of the principal axes really cuts the surface, which is a *double sheeted hyperboloid.* If all three c's are positive, the surface is cut in real points by all three axes, and the quadric is an *ellipsoid.* But if all the c's are negative, there are no real semiaxes, and we get the so-called *imaginary ellipsoid*.

It is well known (Ham. Elem., § 354; Tait, §§ 163, 164) that the three c's are the roots of an algebraic cubic, and are always real. We shall find it convenient to take these roots in the algebraic order: $c_1 < c_2 < c_3$. The general scalar equation,

$$S\varrho\varphi\varrho = 1,$$

where φ is self-conjugate, may be written in the rectangular form: —

$$c_1 S^2 i\varrho + c_2 S^2 j\varrho + c_3 S^2 k\varrho = 1.$$

If any two roots, as c_2 and c_3, are equal, a plane

$$Si\varrho = m$$

perpendicular to the third direction (i) cuts the surface in the circular section,

$$c_2 S^2 j\varrho + c_2 S^2 k\varrho = 1 - c_1 m^2.$$

So the quadric is a surface of revolution. If all the roots are equal — and this, of course, can only happen in the case of an ellipsoid, —

$$c_1 S^2 i\varrho + c_1 S^2 j\varrho + c_1 S^2 k\varrho = 1,$$

and therefore $$S^2 i\varrho + S^2 j\varrho + S^2 k\varrho = \frac{1}{c_1},$$

$$\text{or, } T^2\varrho \left(\cos^2 < \frac{i}{\rho} + \cos^2 < \frac{j}{\rho} + \cos^2 < \frac{k}{\rho}\right) = T^2\varrho = \frac{1}{c_1},$$

and $$T\varrho = \frac{1}{\sqrt{c_1}}.$$

The surface is then a sphere with $c_1^{-\frac{1}{2}}$ for a radius.

If the constant term vanishes, we have already seen that the surface must be a cone. Neither of the principal diameters can be in the direction of a side of the cone; for, if

$$\varrho = xi$$

for instance, we find

$$c_1 S^2 ixi = x^2 c_1 = 0;$$

and therefore $$x = 0.$$

POLES AND POLAR PLANES.

To find the locus of the point of harmonic division of radii through a point α, transfer the origin to that point, and the equation

$$S\varrho\varphi\varrho = 1$$

becomes

$$S\varrho\varphi\varrho + 2\,S\varrho\varphi\alpha + S\alpha\varphi\alpha = 1.$$

Let the vectors of the surface on any line passing through the new origin be ϱ' and ϱ'', and let their harmonic mean be ϱ. We must have then

$$\frac{2}{T\rho} = \frac{1}{T\rho'} + \frac{1}{T\rho''}.$$

If, now, we take ϱ in the direction β, the equation of the surface becomes

$$T\varrho^2 S\beta\varphi\beta + 2T\varrho S\beta\varphi\alpha + S\alpha\varphi\alpha = 1.$$

$T\varrho'$ and $T\varrho''$ are the roots of this quadratic; and applying to it the well-known principles of quadratics, we have

$$\frac{2}{T\rho} = \frac{1}{T\rho'} + \frac{1}{T\rho''} = \frac{T\rho'' + T\rho'}{T\rho' T\rho''} = \frac{-2\,S\beta\phi\alpha}{S\beta\phi\beta} \div \frac{S\alpha\phi\alpha - 1}{S\beta\phi\beta} = \frac{-2\,S\beta\phi\alpha}{S\alpha\phi\alpha - 1};$$

which gives

$$T\varrho S\beta\varphi\alpha = 1 - S\alpha\varphi\alpha;$$

or, since β is the versor of ϱ,

$$S\varrho\varphi\alpha + S\alpha\varphi\alpha = 1;$$

which is the equation of a plane, the polar plane of the new origin. Transfer back to the former origin, by substituting $\varrho - \alpha$ for ϱ, and we find

$$S\varrho\varphi\alpha = 1$$

as the equation of the polar plane of α.

This last equation can be written

$$S\alpha\varphi\varrho = 1,$$

and by changing the variable we get, of course, the polar plane of ϱ. So we see that, if α is on the polar plane of β ($S\alpha\varphi\beta = 1$), β is on the polar plane of α ($S\beta\varphi\alpha = 1$).

We have found that the equation of the tangent plane at a point ϱ is

$$S\omega\varphi\varrho = 1.$$

If we consider ω fixed and ϱ variable, we shall find the equation of a plane containing all the points where tangents from ω meet the surface. But this

$$S\varrho\varphi\omega = 1$$

is the same equation that we have just found for the polar plane of ω. And this is indeed what we should expect, for if two radii vectores from ω become equal, their harmonic mean is equal to each of them, and must reach the surface where they do. We see that the polar plane of any point on a tangent plane must pass through the point of contact, and that the polar plane of any point cuts the surface in the locus of the points of contact of tangent lines drawn from that point.

RELATIONS BETWEEN POLAR PLANES AND CONJUGATE DIAMETERS.

The function $\varphi\varrho$ is, as we have seen, not changed in direction by varying the tensor of ϱ. The polar planes, then, of all points in the same straight line from the origin are parallel, for they are all represented by

$$S\varrho\varphi\alpha = 1,$$

where the tensor only of α varies. But the polar plane

$$S\varrho\varphi\alpha = 1,$$

where α is a vector of the surface, becomes a tangent plane; and this is parallel to

$$S\varrho\varphi\alpha = 0,$$

the diametral plane bisecting all chords parallel to α. Hence, we find that the polar plane of any point is parallel to the diametral plane conjugate to the diameter passing through the point. From another property of conjugate diameters, it is seen that the diameter through any point bisects all chords that it meets parallel to the polar plane of that point. Again,

$$S\alpha\varphi\varrho = 1$$

is the same as

$$S\alpha(\varphi\varrho - \alpha^{-1}) = 0;$$

and, in order that this should be equal to

$$S\alpha\varphi\varrho = 0,$$

α^{-1} must vanish and α become infinite. Thus, in the same way that a tangent plane is the polar plane of a point on the quadric, a diametral plane is the polar plane of a point at infinity.

From the relation between polar planes and conjugate diameters, it is evident that there are three rectangular directions for which the polar plane of a point is perpendicular to the vector from the origin to that point. It is, perhaps, needless to add that in cases where two of the roots c_1, c_2, c_3 are equal, — that is, where the directions for which $\varphi\varrho$ is parallel to ϱ, degenerate into one vector and any two in a plane perpendicular to it, — the directions for which central radii and polar planes are perpendicular degenerate in the same way, and we have surfaces of revolution.

In the central equation of the cone, we have already noticed that the constant term vanishes. If, now, we take the general central equation of the cone in the form

$$S\varrho\varphi\varrho = 0,$$

a tangent plane to the surface at any point α is represented by

$$S\varrho\varphi\alpha = 0;$$

but this is satisfied by

$$\varrho = 0.$$

Every tangent plane to a cone, then, passes through the centre, or vertex. The equation of the polar plane of any point is

$$S\varrho\varphi\alpha = 0,$$

and this also passes through the centre. It is parallel to the diametral plane conjugate to α, and since both of these planes pass through the centre they must coincide. Indeed, both are represented by the same equation

$$S\varrho\varphi\alpha = 0.$$

We see, moreover, that this is the polar plane of all points on the line

$$\varrho = x\alpha,$$

because all such planes are parallel and all pass through the origin.

Cyclic Normals.

I wish to say only a word about these remarkable vectors, in order to show the connection between the equation of the *central quadrics* and the self-conjugate part of that of the *paraboloids*. If we take the cyclic transformation of the equation

$$\begin{aligned} c = S\rho\varphi\rho &= g\rho^2 + S\rho V\lambda\rho\mu \\ &= g\rho^2 + S\rho(-\rho S\lambda\mu + \lambda S\mu\rho + \mu S\lambda\rho) \\ &= \rho^2(g - S\lambda\mu) + 2S\lambda\rho S\mu\rho\,; \end{aligned}$$

and if we cut the surface by planes perpendicular to λ or μ, — *i.e.*, by the planes

$$S\lambda\rho = c',$$

$$S\mu\rho = c'',$$

we find

$$(g - S\lambda\mu)\rho^2 + 2\,c'S\mu\rho = c,$$

$$(g - S\lambda\mu)\rho^2 + 2\,c''S\lambda\rho = c,$$

either of which is a sphere whose intersection with the plane is a circle. This would still be true if c should vanish, and the quadric become a cone; the only difference being that in this case the origin would be on the surface of the sphere.

TANGENT CONE.

The plane passing through all the points of contact of tangents from α to the quadric

$$S\rho\varphi\rho = 1,$$

is the polar plane of α, as we have seen, and its equation is

$$S\rho\varphi\alpha = 1.$$

A surface of the second order tangent to the quadric along its intersection with this plane will be represented by

$$S\rho\varphi\rho - 1 + x(S\rho\varphi\alpha - 1)^2 = 0.$$

If this surface pass through α, its equation must be satisfied by α, and therefore

$$S\alpha\varphi\alpha - 1 + x(S\alpha\varphi\alpha - 1)^2 = 0,$$

and this gives

$$x = \frac{-1}{(S\alpha\phi\alpha - 1)}.$$

Substituting this value of x, we obtain the equation

$$(S\rho\varphi\rho - 1)\,(S\alpha\varphi\alpha - 1) - (S\rho\varphi\alpha - 1)^2 = 0.$$

Transfer the origin to α, and this becomes

$$(S\varrho\varphi\varrho + 2\,S\varrho\varphi\alpha + S\alpha\varphi\alpha - 1)(S\alpha\varphi\alpha - 1) - (S\varrho\varphi\alpha + S\alpha\varphi\alpha - 1)^2$$
$$= S\varrho\varphi\varrho(S\alpha\varphi\alpha - 1) - (S\varrho\varphi\alpha)^2 = 0.$$

This equation represents a cone referred to its centre, because every term contains ϱ to the second degree. It is, then, the *tangent cone* from α.

If in the equation of the polar plane

$$S\varrho\varphi\alpha = 1$$

α vanish, no finite value of ϱ will satisfy the equation, and the polar plane of the origin is seen to lie at infinity. The tangent cone from the origin must therefore meet the quadric at infinity, and becomes what is called the *asymptotic cone.* The equation of this cone is readily obtained by substituting

$$\alpha = 0$$

in that of the tangent cone

$$S\varrho\varphi\varrho(S\alpha\varphi\alpha - 1) - (S\varrho\varphi\alpha)^2 = 0,$$

which gives

$$S\varrho\varphi\varrho = 0.$$

The rectangular transformation for this is

$$c_1 S^2\alpha_1\varrho + c_2 S^2\alpha_2\varrho + c_3 S^2\alpha_3\varrho = 0,$$

and it can only be satisfied by real finite values of ϱ, where one or more c's are negative, and the quadric an *hyperboloid.* In the case of the real or imaginary *ellipsoid*, the c's are all of the same sign, and the asymptotic cone is reduced to its vertex, for its equation is only satisfied by

$$\varrho = 0.$$

The reality of any cone depends of course, in the same way, on the difference of the signs of the roots of $\varphi\varrho$. Any cone may, then, be regarded as the limiting case of an hyperboloid which is degenerating into its own asymptotic cone.

This idea leads us to consider how one form of surface of the second order may, by the modification of the constants in its equation, pass by imperceptible degrees into some different form.

When any root of a central quadric vanishes, the surface becomes indeterminate in the direction of that root, and thus degenerates into a cylinder. If c_1, for instance, vanishes, we have

$$c_2 S^2\alpha_2\varrho + c_3 S^2\alpha_3\varrho = 1,$$

and this represents a *cylinder;* because to any radius vector ρ we can add $x\alpha_1$, where x is any scalar. But what is to become of the asymptotic cone in this case? It must also be indeterminate in the same direction, and yet it must still retain the property that all radii vectores must lie wholly in its surface. The only surfaces of the second order of which this can be true are pairs of real or imaginary planes.

The quaternion expression for this is very interesting. If the negative root of a single-sheeted hyperboloid or either root of an ellipsoid vanishes, the quadric is represented by the equation

$$c_2 S^2\alpha_2\rho + c_3 S^2\alpha_3\rho = 1,$$

and becomes an elliptic cylinder. The asymptotic cone

$$c_2 S^2\alpha_2\rho + c_3 S^2\alpha_3\rho = 0,$$

or

$$c_2 S^2\alpha_2\rho = - c_3 S^2\alpha_3\rho,$$

or

$$\sqrt{c_2}\, S\alpha_2\rho = \pm \sqrt{- c_3}\, S\alpha_3\rho,$$

becomes a pair of imaginary planes, containing only one real line,

$$\rho = x\alpha_1,$$

the line of their intersection, which satisfies the equation because it makes both sides vanish. This may be considered a sort of intermediate case, on a roundabout road, between the real (hyperbolic) asymptotic cone and the imaginary (elliptic) one.

If the positive root of a double-sheeted hyperboloid vanish, the surface

$$- c_1 S^2\alpha_1\rho - c_2 S^2\alpha_2\rho = 1$$

becomes an imaginary elliptic cylinder. The asymptotic cone

$$- c_1 S^2\alpha_1\rho - c_2 S^2\alpha_2\rho = 0,$$

or

$$\sqrt{- c_1}\, S\alpha_1\rho = \pm \sqrt{c_2}\, S\alpha_2\rho,$$

again represents a pair of imaginary planes, containing only one real line,

$$\rho = x\alpha_3,$$

this time at right angles to the former direction.

If, finally, a positive root of a single-sheeted hyperboloid or a negative root of a double-sheeted one vanish, so as to leave the two actual roots with opposite signs, the quadric degenerates into

$$c_1 S^2\alpha_1\varrho - c_3 S^2\alpha_3\varrho = 1,$$

an hyperbolic cylinder. In this case, the asymptotic cone is

$$c_1 S^2\alpha_1\varrho - c_3 S^2\alpha_3\varrho = 0,$$

$$c_1 S^2\alpha_1\varrho = c_3 S^2\alpha_3\varrho,$$

or

$$\sqrt{c_1} S\alpha_1\varrho = \pm \sqrt{c_3} S\alpha_3\varrho,$$

a pair of real planes tangent to the cylinder at infinity.

When two roots vanish,

$$c_3 S^2\alpha_3\varrho = 1$$

represents two parallel planes, real or imaginary, according as the actual root is positive or negative. The asymptotic cone

$$c_3 S^2\alpha_3\varrho = 0,$$

or

$$S\alpha_3\varrho = 0,$$

is a plane — which we might call a double plane — passing through the origin and parallel to the pair of planes. These cases of degeneracy of quadrics we are about to study from an entirely different point of view.

NON-CENTRAL QUADRICS.

It has been already proved (page 224) that the centre of a quadric is found by solving for δ the equation

$$\varphi_0\delta + \gamma = 0.$$

Now the self-conjugate function may be treated under several different forms, such as the *rectangular*, *cyclic*, or *focal* (Ham. Elem., § 359, I., III., and V.). Of these, I shall, for the sake of generality, consider two, the *rectangular* and the *cyclic*, although the former is far more convenient than the latter.

To solve the equation

$$\varphi_0\delta + \gamma = 0,$$

I shall use the general formula (Ham., §§ 347–350; Tait, Chap. V.)

$$m\varrho = m\varphi^{-1}\gamma = m'\gamma - m''\varphi\gamma + \varphi^2\gamma,$$

where

$$m = \frac{S\phi'\lambda\phi'\mu\phi'\nu}{S\lambda\mu\nu},$$

$$m' = \frac{S(\lambda\phi'\mu\phi'\nu + \phi'\lambda.\mu.\phi'\nu + \phi'\lambda\phi'\mu.\nu)}{S\lambda\mu\nu},$$

and

$$m'' = \frac{S(\lambda\mu\phi'\nu + \lambda\phi'\mu.\nu + \phi'\lambda.\mu.\nu)}{S\lambda\mu\nu}.$$

Since the function we are considering is self-conjugate,

$$\varphi_0 = \varphi_0'.$$

Let us take first the *rectangular* form

$$\varphi_0\delta = c_1\alpha_1 S\alpha_1\delta + c_2\alpha_2 S\alpha_2\delta + c_3\alpha_3 S\alpha_3\delta = -\gamma;$$

and let

$$\alpha_1 = \lambda,\ \alpha_2 = \mu, \text{ and } \alpha_3 = \nu,$$

then

$$\varphi\lambda = \varphi\alpha_1 = -c_1\alpha_1,$$

$$\varphi\mu = \varphi\alpha_2 = -c_2\alpha_2,$$

$$\varphi\nu = \varphi\alpha_3 = -c_3\alpha_3.$$

Substituting these values in our formulas,

$$m = \frac{-c_1c_2c_3 S\alpha_1\alpha_2\alpha_3}{S\alpha_1\alpha_2\alpha_3} = -c_1c_2c_3,$$

$$m' = \frac{S(\alpha_1.c_2\alpha_2.c_3\alpha_3 + c_1\alpha_1.\alpha_2.c_3\alpha_3 + c_1\alpha_1.c_2\alpha_2.\alpha_3)}{S\alpha_1\alpha_2\alpha_3} = c_2c_3 + c_1c_3 + c_1c_2,$$

$$m'' = \frac{S(-\alpha_1.\alpha_2.c_3\alpha_3 - \alpha_1.c_2\alpha_2.\alpha_3 - c_1\alpha_1.\alpha_2.\alpha_3)}{S\alpha_1\alpha_2\alpha_3} = -c_3 - c_2 - c_1.$$

And we find for the general equation of solution

$$m\delta = -c_1c_2c_3\delta = -(c_1c_2 + c_1c_3 + c_2c_3)\gamma - (c_1 + c_2 + c_3)(c_1\alpha_1 S\alpha_1\gamma \\ + c_2\alpha_2 S\alpha_2\gamma + c_3\alpha_3 S\alpha_3\gamma) + c_1^2\alpha_1 S\alpha_1\gamma + c_2^2\alpha_2 S\alpha_2\gamma + c_3^2\alpha_3 S\alpha_3\gamma$$

$$= -(c_1c_2 + c_1c_3 + c_2c_3)\gamma - (c_1c_2 + c_1c_3)\alpha_1 S\alpha_1\gamma \\ - (c_2c_3 + c_2c_1)\ \alpha_2 S\alpha_2\gamma - (c_3c_1 + c_3c_2)\alpha_3 S\alpha_3\gamma;$$

and for the complete solution

$$\delta = +\left(\frac{1}{c_3} + \frac{1}{c_2} + \frac{1}{c_1}\right)\gamma + \left(\frac{1}{c_3} + \frac{1}{c_2}\right)\alpha_1 S\alpha_1\gamma + \left(\frac{1}{c_1} + \frac{1}{c_3}\right)\alpha_2 S\alpha_2\gamma \\ + \left(\frac{1}{c_2} + \frac{1}{c_1}\right)\alpha_3 S\alpha_3\gamma.$$

This will evidently give a single finite value for δ, unless one of the c's vanish. Take the most general case. Let γ be in any direction, such as

$$\gamma = - d\alpha_1 - g\alpha_2 - k\alpha_3.$$

Then

$$S\alpha_1\gamma = - dS\alpha_1^2 = + d,$$

$$S\alpha_2\gamma = + g, \text{ and } S\alpha_3\gamma = + k,$$

$$\delta = \left(\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}\right)(- d\alpha_1 - g\alpha_2 - k\alpha_3) + \left(\frac{1}{c_2} + \frac{1}{c_3}\right) d\alpha_1 + \left(\frac{1}{c_3} + \frac{1}{c_1}\right) g\alpha_2 + \left(\frac{1}{c_1} + \frac{1}{c_2}\right) k\alpha_3$$

$$= - \frac{1}{c_1} d\alpha_1 - \frac{1}{c_2} g\alpha_2 - \frac{1}{c_3} k\alpha_3.$$

Then if one root, say c_1, vanishes, the centre is at an infinite distance in the direction of α_1, but at a finite distance in each of the other directions. Now let us substitute

$$c_1 = 0,$$

and

$$\gamma = - d\alpha_1 - g\alpha_2 - k\alpha_3$$

in the general equation of the second degree

$$S\rho\phi\rho + 2\, S\gamma\rho = c = c_1S^2\alpha_1\rho + c_2S^2\alpha_2\rho + c_3S^2\alpha_3\rho + 2\, S\gamma\rho.$$

We find

$$c_2S^2\alpha_2\rho + c_3S^2\alpha_3\rho - 2\, dS\alpha_1\rho - 2\, gS\alpha_2\rho - 2\, kS\alpha_3\rho = c.$$

We found that $\frac{1}{c_2}g$ and $\frac{1}{c_3}k$ were the distances of the centre in the directions of α_2 and α_3. To bring the origin into a line with the centre in these two directions, substitute

$$\rho = \rho - \frac{g}{c_2}\alpha_2 - \frac{k}{c_3}\alpha_3,$$

and the equation takes the form

$$c_2S^2\alpha_2\rho + c_3S^2\alpha_3\rho - 2\, dS\alpha_1\rho = c.$$

This may be still farther simplified, by taking for origin that point where α_1 meets the surface. Let

$$\rho = x\alpha_1,$$

$$-2\,dS\alpha_1 x\alpha_1 = +x\,2\,d = c,$$

$$x = \frac{+c}{2\,d}.$$

Now substitute

$$\rho = \rho + \frac{c}{2\,d}\,\alpha_1,$$

and we find

$$c_2 S^2\alpha_2\rho + c_3 S^2\alpha_3\rho - 2\,dS\alpha_1\rho + 2\,d\,\frac{c}{2\,d} = c,$$

$$c_2 S^2\alpha_2\rho + c_3 S^2\alpha_3\rho - 2\,dS\alpha_1\rho = 0.$$

The quadric represented by this equation is an *elliptic* or *hyperbolic paraboloid*, according as the c's are alike or unlike in sign. Because if the surface be cut by a plane

$$S\alpha_2\rho = c \text{ or } S\alpha_3\rho = c$$

perpendicular to either α_2 or α_3, the section is

$$c_2 S^2\alpha_2\rho = 2\,dS\alpha_1\rho - c,$$

or

$$c_3 S^2\alpha_3\rho = 2\,dS\alpha_1\rho - c,$$

either of which is easily seen to be a *parabola*. But if cut by a plane perpendicular to α_1

$$S\alpha_1\rho = c,$$

the section is

$$c_2 S^2\alpha_2\rho + c_3 S^2\alpha_3\rho = -c,$$

which is an *ellipse* or *hyperbola*, according as c_2 and c_3 are alike or unlike in sign. The sign of c shows the side of the origin in which the cutting plane lies, and determines whether the elliptic section is real or imaginary, or whether the hyperbolic section has its transverse axis parallel to α_2 or to α_3.

We have considered the case in which

$$c_1 = 0,$$

and

$$\gamma = -d\alpha_1 - g\alpha_2 - k\alpha_3.$$

Let us now go a step farther, and suppose d, g, or k to vanish, and let us first take it as the one corresponding to the root that has disappeared. In this case

$$d = 0;$$

and therefore

$$\gamma = -g\alpha_2 - k\alpha_3.$$

The vector of the centre has been found to be

$$\delta = -\frac{d}{c_1}\,\alpha_1 - \frac{g}{c_2}\,\alpha_2 - \frac{k}{c_3}\,\alpha_3.$$

But, if both d and c_1 vanish, the centre must be at a determinate finite distance in two directions, and at an indeterminate distance in the other. The general equation

$$S\rho\varphi_0\rho + 2\,S\gamma\rho = c$$

becomes

$$c_2 S^2\alpha_2\rho + c_3 S^2\alpha_3\rho - 2\,gS\alpha_2\rho - 2\,kS\alpha_3\rho = c.$$

Reduce this to some point in the central line, by substituting

$$\rho = \rho - \frac{g}{c_2}\,\alpha_2 - \frac{k}{c_3}\,\alpha_3,$$

and our equation takes the form

$$c_2 S^2\alpha_2\rho + c_3 S^2\alpha_3\rho = c.$$

This represents a cylinder, because we may add $x\alpha_1$ to any vector without affecting the equation; as, indeed, we can see by inspection of the unreduced form of equation for this case. We have thus found again, by a totally different process, the same case of degeneracy considered on pages 235, 236, and 237. And in comparing our results we must remember that the function there called $\varphi\rho$, and the one called $\varphi_0\rho$ here, are exactly the same; at least, with the exception that the former was multiplied by a constant, in order to make the constant term of the equation equal to unity.

If, in the equation for the cylinder,

$$c_2 S^2\alpha_2\rho + c_3 S^2\alpha_3\rho - 2\,gS\alpha_2\rho - 2\,kS\alpha_3\rho = c,$$

$$c = -\frac{g^2}{c_2} - \frac{k^2}{c_3},$$

the equation is a complete square, and equivalent to

$$\left(\sqrt{c_2}\,S\alpha_2\rho - \frac{g}{\sqrt{c_2}}\right) = \pm\left(\sqrt{c_3}\,S\alpha_3\rho - \frac{k}{\sqrt{c_3}}\right).$$

And this represents a pair of real planes, or a pair of imaginary planes with the real intersection

$$\rho = x\alpha_1,$$

according as c_2 and c_3 have like or opposite signs. But these conditions for the value of c are really the same that we found before, in order that a quadric should degenerate into its asymptotic cone, and this is because if with this value of c we transfer the origin to the centre,

$$\delta = -\frac{g}{c_2}\alpha_2 - \frac{k}{c_3}\alpha_3,$$

we obtain an equation without any absolute term. If in the equation

$$\gamma = -d\alpha_1 - g\alpha_2 - k\alpha_3$$

g or k vanish, when c_1 alone of the roots disappears, it would merely indicate that the origin was already in a line with the centre as far as that direction is concerned.

Suppose two roots, as c_1 and c_2, vanish, and also the co-efficient of γ in the direction of one of them, say d. The solution for the centre will then give

$$\delta = -\frac{1}{c_1}d\alpha_1 - \frac{1}{c_2}g\alpha_2 - \frac{1}{c_3}k\alpha_3.$$

The centre is thus at an infinite distance in one direction, and at an indeterminate distance in another. This can only be true of a *parabolic cylinder*. Let us see what the general equation of the second degree will give us in this case.

$$c_1S^2\alpha_1\varrho + c_2S^2\alpha_2\varrho + c_3S^2\alpha_3\varrho + 2\,S\gamma\varrho = c$$

becomes

$$c_3S^2\alpha_3\varrho - 2\,gS\alpha_2\varrho - 2\,kS\alpha_3\varrho = c.$$

And by cutting this by planes perpendicular to α_1, α_2, and α_3, it can at once be proved to be a *parabolic cylinder*. By substituting

$$\varrho = \varrho - \frac{k}{c_3}\alpha_3 + \frac{c + \frac{2\,k^2}{c_3}}{2\,g}\alpha_2,$$

the equation can be reduced to its simplest form

$$c_3S^2\alpha_3\varrho - 2\,gS\alpha_2\varrho = 0.$$

If, as a farther condition, we had

$$g = 0,$$

this last transformation would not be possible; but the equation could be reduced to the form

$$c_3S^2\alpha_3\varrho = c,$$

or

$$\sqrt{c_3}S\alpha_3\varrho = \pm\sqrt{c},$$

and would represent a *pair of planes* perpendicular to α_3, and hence parallel to each other.

For the sake of generality, I shall solve the equation for the centre, and examine a couple of cases arising under it, by means of the *cyclic* transformation. And although the result is far from satisfactory, yet the solution contains some points of interest, and is worth inserting in spite of its length. The well-known formula for this form of transformation is (Ham. Elem., § 357 (5)and (8)):

$$\varphi_0\varrho = g\varrho + \mathrm{V}\lambda\varrho\mu = (g - \mathrm{S}\lambda\mu)\varrho + \lambda\mathrm{S}\mu\varrho + \mu\mathrm{S}\lambda\varrho,$$

where it is found that if we take $c_1 < c_2 < c_3$,

$$c_2 = -g + \mathrm{S}\lambda\mu,$$

$$c_1 = -g - \mathrm{T}\lambda\mu,$$

$$c_3 = -g + \mathrm{T}\lambda\mu,$$

$$\alpha_1 = \mathrm{U}(\lambda\mathrm{T}\mu - \mu\mathrm{T}\lambda),$$

$$\alpha_2 = \mathrm{UV}\lambda u,$$

$$\alpha_3 = \mathrm{U}(\lambda\mathrm{T}\mu + \mu\mathrm{T}\lambda).$$

To solve

$$g\varrho + \mathrm{V}\lambda\varrho\mu = \gamma$$

(changing for the sake of convenience the sign of γ) by means of the formulas of page 238. Let

$$\lambda = \lambda,\ \mu = \mu, \text{ and } \nu = \gamma.$$

Here, again, φ is, of course, self-conjugate as before. Now we find

$$\varphi\lambda = g\lambda + \lambda^2\mu,$$

$$\varphi\mu = g\mu + \lambda\mu^2,$$

$$\varphi\nu = g\gamma + \mathrm{V}\lambda\gamma\mu.$$

And substituting these values: —

$$\begin{aligned}\mathrm{S}\varphi\lambda\varphi\mu\varphi\nu &= \mathrm{S}(g\lambda + \lambda^2\mu)(g\mu + \lambda\mu^2)(g\gamma + \lambda\mathrm{S}\gamma\mu - \gamma\mathrm{S}\lambda\mu + \mu\mathrm{S}\lambda\gamma)\\ &= \mathrm{S}(g^2\lambda\mu + 2\,g\lambda^2\mu^2 + \mu^3\lambda^3)(g\gamma + \lambda\mathrm{S}\gamma\mu - \gamma\mathrm{S}\lambda\mu + \mu\mathrm{S}\lambda\gamma)\\ &= \mathrm{S}(g^3\lambda\mu\gamma + 2\,g^2\lambda^2\mu^2\gamma - g\lambda^3\mu^3\gamma + g^2\lambda^2\mu\mathrm{S}\gamma\mu + 2\,g\lambda^3\mu^2\mathrm{S}\gamma\mu\\ &\quad + \lambda^4\mu^3\mathrm{S}\gamma\mu - g^2\lambda\mu\gamma\mathrm{S}\lambda\mu - 2\,g\lambda^2\mu^2\gamma\mathrm{S}\lambda\mu - \mu^3\lambda^3\gamma\mathrm{S}\lambda\mu + \text{vector terms})\\ &= \mathrm{S}(g^3.\lambda\mu\gamma - g\lambda^2\mu^2.\lambda\mu\gamma - g^2\lambda\mu\gamma.\mathrm{S}\lambda\mu + \lambda\mu\gamma.\lambda^2\mu^2\mathrm{S}\lambda\mu).\end{aligned}$$

From which

$$m = g^3 - g\lambda^2\mu^2 - g^2 S\lambda\mu + \lambda^2\mu^2 S\lambda\mu,$$

$$= (g - S\lambda\mu)(g^2 - \lambda^2\mu^2).$$

To find m'

$$S(\lambda\varphi\mu\varphi\nu) = S(\lambda.(g\mu + \mu^2\lambda)(g\gamma + \lambda S\gamma\mu - \gamma S\lambda\mu + \mu S\lambda\gamma))$$

$$= S(\lambda.(g^2\mu\gamma + g\mu\lambda.S\gamma\mu - g\mu\gamma S\lambda\mu + g\mu^2 S\lambda\gamma + g\mu^2\lambda\gamma + \mu^2\lambda^2 S\lambda\mu$$
$$- \mu^2\lambda\gamma S\lambda\mu + \lambda\mu^3 S\lambda\gamma))$$

$$= S(g^2\lambda\mu\gamma - gS\lambda\mu.\lambda\mu\gamma)$$

$$= g^2 S\lambda\mu\gamma - gS\lambda\mu.S\lambda\mu\gamma.$$

In the same way

$$S(\varphi\lambda.\mu.\varphi\nu) = g^2 S\lambda\mu\gamma - gS\lambda\mu.S\lambda\mu\gamma.$$

But

$$S(\varphi\lambda.\varphi\mu.\nu) = S(g\lambda + \lambda^2\mu)(g\mu + \mu^2\lambda)\gamma$$

$$= S(g^2\lambda\mu + 2\ g\lambda^2\mu^2 + \mu^3\lambda^3)\gamma$$

$$= g^2 S\lambda\mu\gamma - \lambda^2\mu^2 S\lambda\mu\gamma.$$

Hence

$$m' = 2\ g^2 - 2\ gS\lambda\mu + g^2 - \lambda^2\mu^2 = 3\ g^2 + 2\ gS\lambda\mu - \lambda^2\mu^2.$$

To find m'':—

$$S\varphi\lambda.\mu.\nu = S(g\lambda + \lambda^2\mu)\mu.\gamma = gS\lambda\mu\gamma$$

$$S\lambda.\varphi\mu.\nu = S\lambda(g\mu + \mu^2\lambda)\gamma = gS\lambda\mu\gamma.$$

But

$$S\lambda.\mu.\varphi\nu = S(\lambda.\mu.(g\gamma + \lambda S\mu\gamma - \gamma S\lambda\mu + \mu S\lambda\gamma))$$

$$= S(g\lambda\mu\gamma - S\lambda\mu.\lambda\mu\gamma) = (g - S\lambda\mu)S\lambda\mu\gamma.$$

And therefore,

$$m'' = g + g + g - S\lambda\mu = 3\ g - S\lambda\mu.$$

To find $\varphi^2\gamma$,

$$\varphi\gamma = g\gamma + \lambda S\mu\gamma - \gamma S\lambda\mu + \mu S\lambda\gamma = g\gamma + V\lambda\gamma\mu;$$

hence

$$\varphi^2\gamma = \varphi\varphi\gamma = g(g\gamma + V\lambda\gamma\mu) + V(\lambda(g\gamma + V\lambda\gamma\mu)\mu),$$

$$= g^2\gamma + 2gV\lambda\gamma\mu + V(\lambda V\lambda\gamma\mu.\mu)$$

$$= g^2\gamma + 2\ g(\lambda S\mu\gamma - \gamma S\lambda\mu + \mu S\lambda\gamma) + \mu\lambda^2 S\mu\gamma$$
$$+ \lambda\mu^2 S\lambda\gamma - \lambda S\lambda\mu S\mu\gamma + \gamma S^2\lambda\mu - \mu S\lambda\mu S\lambda\gamma.$$

To find

$$m''\varphi\gamma = (3g - \mathrm{S}\lambda\mu)(g\gamma + \lambda\mathrm{S}\mu\gamma - \gamma\mathrm{S}\lambda\mu + \mu\mathrm{S}\lambda\gamma),$$
$$= 3\, g^2\gamma + 3\, g\lambda\mathrm{S}\mu\gamma - 3\, g\gamma\mathrm{S}\lambda\mu + 3\, g\mu\mathrm{S}\lambda\gamma - g\gamma\mathrm{S}\lambda\mu$$
$$- \lambda\mathrm{S}\lambda\mu\mathrm{S}\mu\gamma + \gamma\mathrm{S}^2\lambda\mu - \mu\mathrm{S}\lambda\mu\mathrm{S}\lambda\gamma.$$

We have then

$$m\varphi^{-1}\gamma = m\delta = m'\gamma - m''\varphi\gamma + \varphi^2\gamma = 3\, g^2\gamma - 2\, g\gamma\mathrm{S}\lambda\mu - \gamma\lambda^2\mu^2 -$$
$$3\, g^2\gamma - 3\, g\lambda\mathrm{S}\mu\gamma + 4\, g\gamma\mathrm{S}\lambda\mu - 3\, g\mu\mathrm{S}\lambda\gamma + g^2\gamma + 2\, g\lambda\mathrm{S}\mu\gamma$$
$$- 2\, g\gamma\mathrm{S}\lambda\mu + 2\, g\mu\mathrm{S}\lambda\gamma + \mu\lambda^2\mathrm{S}\mu\gamma + \lambda\mu^2\mathrm{S}\lambda\gamma.$$
$$= - \gamma\lambda^2\mu^2 - g\lambda\mathrm{S}\mu\gamma - g\mu\mathrm{S}\lambda\gamma + g^2\gamma + \mu\lambda^2\mathrm{S}\mu\gamma + \lambda\mu^2\mathrm{S}\lambda\gamma$$
$$= \lambda(\mu^2\mathrm{S}\lambda\gamma - g\mathrm{S}\mu\gamma) + \mu(\lambda^2\mathrm{S}\mu\gamma - g\mathrm{S}\lambda\gamma) + \gamma(g^2 - \lambda^2\mu^2).$$

And the complete solution for the centre is

$$\delta = \frac{\lambda(\mu^2\mathrm{S}\lambda\gamma - g\mathrm{S}\mu\gamma) + \mu(\lambda^2\mathrm{S}\mu\gamma - g\mathrm{S}\lambda\gamma) + \lambda(g - \mathrm{T}\lambda\mu)(g + \mathrm{T}\lambda\mu)}{(g - \mathrm{S}\lambda\mu)(g^2 - \lambda^2\mu^2) = (g - \mathrm{S}\lambda\mu)(g - \mathrm{T}\lambda\mu)(g + \mathrm{T}\lambda\mu)}.$$

From the values of c_1, c_2, and c_3 given on page 243, we see that, as we found before under the *rectangular* form, δ has a single finite value, unless one of the roots c_1, c_2, or c_3 of $\varphi_0\varrho$ vanishes. This *cyclic* solution for δ, because it contains no explicit rectangular vectors, is difficult to use; and, indeed, often assumes a hopelessly indeterminate form.

We can, however, obtain with sufficient ease in this cyclic form the equations of the *paraboloids.* It was found under the rectangular transformation that, in order that the equation of the second degree should represent a paraboloid, we must have c_2 or c_1 disappear; that is,

$$g = \mathrm{S}\lambda\mu \text{ or } g = - \mathrm{T}\lambda\mu.$$

Let us consider the latter case. We know (Ham., § 357 (9), XXII.)

$$\mathrm{S}\lambda\varrho\mu\varrho - \varrho^2\mathrm{T}\lambda\mu = \{(\mathrm{S}\lambda\mu\varrho)^2 + (\mathrm{S}\lambda\varrho.\mathrm{T}\mu + \mathrm{S}\mu\varrho\mathrm{T}\lambda)^2\} \times (\mathrm{T}\lambda\mu -$$
$$\mathrm{S}\lambda\mu)^{-1} = 2\, \mathrm{S}\gamma\varrho + c,$$

if $$g = - \mathrm{T}\lambda\mu;$$

for $$g\varrho^2 + \mathrm{S}\lambda\varrho\mu\varrho = 2\, \mathrm{S}\gamma\varrho + c$$

is a form of the general equation of the second degree, and therefore

$$(\mathrm{S}\lambda\mu\varrho)^2 + (\mathrm{S}\lambda\varrho\mathrm{T}\mu + \mathrm{S}\mu\varrho\mathrm{T}\lambda)^2 = (\mathrm{T}\lambda\mu - \mathrm{S}\lambda\mu)(2\, \mathrm{S}\gamma\varrho + c).$$

It is evident from the forms of the terms of the first member that the resolved parts of γ in the rectangular directions $\mathrm{UV}\lambda\mu$ and $\mathrm{U}(\lambda\mathrm{T}\mu + \mu\mathrm{T}\lambda)$ can be eliminated by taking a new origin; and the γ reduced to the third direction $\mathrm{U}(\lambda\mathrm{T}\mu - \mu\mathrm{T}\lambda)$.

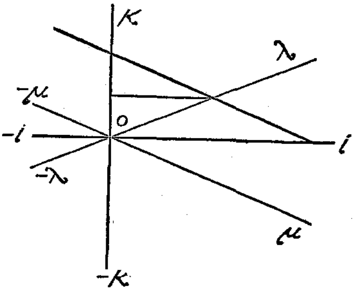


FIG. 3.

Now
$$\mathrm{T}(\lambda\mathrm{T}\mu - \mu\mathrm{T}\lambda) = \mathrm{T}\lambda\mu\mathrm{T}(\mathrm{U}\lambda - \mathrm{U}\mu)$$
$$= 2\,\mathrm{T}\lambda\mu \sin \tfrac{1}{2} <^{\lambda}_{\mu} = \sqrt{2\,\mathrm{T}^2\lambda\mu(1 - \cos <^{\lambda}_{\mu})},$$
$$= \sqrt{2}\,(\mathrm{T}^2\lambda\mu + \mathrm{T}\lambda\mu\,\mathrm{S}\lambda\mu)^{\frac{1}{2}}.$$

If k be the length of γ, our equation becomes
$$(\mathrm{S}\lambda\mu\rho)^2 + (\mathrm{S}\lambda\rho\mathrm{T}\mu + \mathrm{S}\mu\rho\mathrm{T}\lambda)^2$$
$$= \sqrt{2}\,k(\mathrm{T}\lambda\mu - \mathrm{S}\lambda\mu)^{\frac{1}{2}}\,\mathrm{T}^{-\frac{1}{2}}\lambda\mu(\mathrm{S}\lambda\rho\mathrm{T}\mu - \mathrm{S}\mu\rho\mathrm{T}\lambda + c)$$
$$= \sqrt{2}.k\mathrm{T}^{-\frac{1}{2}}\lambda\mu(\mathrm{T}\lambda\mu - \mathrm{S}\lambda\mu)^{\frac{1}{2}}\{\mathrm{S}(\lambda\rho\mathrm{T}\mu - \mu\rho\mathrm{T}\lambda) + c\}.$$

This is an *elliptic paraboloid;* for, if cut by a plane perpendicular to $\mathrm{V}\lambda\mu$, or $\mathrm{U}(\lambda\mathrm{T}\mu + \mu\mathrm{T}\lambda)$, the section is a *parabola.* But, if by a plane perpendicular to $\mathrm{U}(\lambda\mathrm{T}\mu - \mu\mathrm{T}\lambda)$, it is an *ellipse.*

To obtain the *hyperbolic paraboloid,* let
$$c_2 = 0, \quad g = \mathrm{S}\lambda\mu.$$

The general cyclic form of the equation of the second degree, in this case, reduces to
$$\mathrm{S}\lambda\rho\mathrm{S}\mu\rho = \mathrm{S}\gamma\rho + c.$$

By a transfer of the origin in the plane of λ and μ, γ may be reduced to a direction perpendicular to that plane; and our equation to the form
$$\mathrm{S}\lambda\rho\mathrm{S}\mu\rho = n\mathrm{S}\lambda\mu\rho + c.$$

Cut this by a plane perpendicular to $V\lambda\mu$, that is, by the plane

$$S\lambda\mu\rho = c',$$

and we have

$$S\lambda\rho S\mu\rho = c'',$$

which is an *hyperbola*. If now the surface be cut by planes perpendicular to one of the other axes $(\lambda + \mu)$ or $(\lambda - \mu)$, the planes

$$S\rho(\lambda + \mu) = 0$$

or

$$S\rho(\lambda - \mu) = 0$$

our equation gives

$$\pm\, S^2\lambda\rho = S\lambda\mu\rho + c.$$

And these are both *parabolas*. These equations for paraboloids can be verified by substituting in the cyclic forms the value of λ and μ in terms of c_1, c_2, and c_3, α_1, α_2, and α_3, given in Hamilton's Elements of Quaternions, § 357, XX., and seq.

In the general equation in δ, for finding the centre

$$(g - S\lambda\mu)\delta + \lambda S\mu\delta + \mu S\lambda\delta = \gamma,$$

if c_1 and c_2 both vanish, and

$$\gamma = di + hk,$$

we have

$$-\,T\lambda\mu = S\lambda\mu\,;$$

whence

$$1 = \cos \angle{}^{\lambda}_{\mu}$$

λ is then parallel to μ, and therefore

$$2\lambda S\lambda\delta = di + hk.$$

The centre must be at a finite distance in the direction of λ, which is the same as that of i, at an infinite distance in the direction of k, and indeterminate in that of j; since we may add xj to δ without affecting the equation. The surface must be a *parabolic cylinder*. Now, if h also vanish, k will be in the same condition as j, and δ will be anywhere in a certain plane perpendicular to i. Let us study this case a moment, for it is the simplest of the non-central quadrics under the cyclic form. The general cyclic equation of the second degree

$$(g - S\lambda\mu)\rho^2 + 2\, S\lambda\rho S\mu\rho = 2\, S\gamma\rho + c,$$

in this case, assumes the form

$$S^2\lambda\rho = hS\lambda\rho + c.$$

But this can be reduced to the form

$$S^2\lambda\varrho - S\lambda\beta S\lambda\varrho - S\lambda\delta S\lambda\varrho + S\lambda\beta S\lambda\delta = 0,$$

for we only assume

$$h = S\lambda\beta + S\lambda\delta$$

and

$$c = - S\lambda\beta S\lambda\delta,$$

and these are but two conditions to determine two unknown quantities. Our equation now becomes

$$(S\lambda\varrho - S\lambda\beta)(S\lambda\varrho - S\lambda\delta) = 0,$$

and may be decomposed into

$$S\lambda(\varrho - \beta) = 0$$

and

$$S\lambda(\varrho - \delta) = 0.$$

Two parallel planes perpendicular to λ.

CIRCULAR SECTIONS.

Almost the only things worthy of notice about the paraboloids are their planes of circular sections; and these are interesting chiefly on account of their connection with the planes of circular sections of the central quadrics.

Differentiating the equation of the paraboloid

$$c_2S^2\alpha_2\varrho + c_3S^2\alpha_3\varrho = 2kS\alpha_1\varrho,$$

we obtain that of its tangent plane

$$2\ c_2S\alpha_2\varrho S\alpha_2\varrho' + 2\ c_3S\alpha_3\varrho S\alpha_3\varrho' = 2kS\alpha_1\varrho'.$$

At the origin, ϱ vanishes, and dropping the accent of ϱ', we get for the tangent plane

$$S\alpha_1\varrho = 0.$$

Now if we find a sphere with the same tangent plane, obtain the equation of the surface of the second order in which lies the intersection of this sphere with the paraboloid, and finally discover the condition that this surface should degenerate into a pair of planes, we shall have the planes whose intersection with the sphere, and consequently with the paraboloid, will be circles.

Such a sphere will be represented by the equation

$$\rho^2 = 2n\mathrm{S}\alpha_1\rho,$$

for this has

$$\mathrm{S}\alpha_1\rho = 0$$

for a tangent plane at the origin. Its intersection with the paraboloid is given by

$$c_2\mathrm{S}^2\alpha_2\rho + c_3\mathrm{S}^2\alpha_3\rho - \frac{k}{n}\,\rho^2 = 0\,;$$

a cone referred to its vertex. The cone will represent a pair of planes, if

$$c_2 = \frac{-k}{n}\,;$$

for then

$$c_2(\mathrm{S}^2\alpha_2\rho + \rho^2) + c_3\mathrm{S}^2\alpha_3\rho = 0.$$

Now if we reduce to Cartesian co-ordinates by the substitution

$$\rho = x\alpha_1 + y\alpha_2 + z\alpha_3,$$

we find that

$$c_2(y^2 - x^2 - y^2 - z^2) + c_3z^2 = 0,$$

$$- c_2(x^2 + z^2) = - c_3z^2\,;$$

and finally

$$c_2x^2 = (c_3 - c_2)z^2.$$

Since we can always take $c_2 < c_3$, — whatever these roots may be, — the equation just found will be imaginary if c_2 and c_3 have opposite signs. The geometrical interpretation of which is that an hyperbolic paraboloid can have no circular section: a fact almost self-evident.

Let us substitute in the equation

$$c_2x^2 = (c_3 - c_2)z^2,$$

the cyclic values of c_1, c_2, and c_3 (Ham. Elem., § 357, XXI.),

$$c_2 - c_1 = \mathrm{T}\lambda\mu + \mathrm{S}\lambda\mu$$

$$c_3 - c_2 = \mathrm{T}\lambda\mu - \mathrm{S}\lambda\mu.$$

In the present case, $c_1 = 0$; and therefore

$$x^2 = \frac{\mathrm{T}\lambda\mu - \mathrm{S}\lambda\mu}{\mathrm{T}\lambda\mu + \mathrm{S}\lambda\mu}\,z^2 = \frac{1 - \cos <^{\lambda}_{\mu}}{1 + \cos <^{\lambda}_{\mu}}\,z^2 = z^2 \tan^2 \tfrac{1}{2} <^{\lambda}_{\mu}$$

$$= z^2 \cot^2 (\tfrac{1}{2} \circlearrowright + \tfrac{1}{2} <^{\lambda}_{\mu}).$$

Now this represents two lines perpendicular to λ and μ

$$x = \pm z \cot\left(\tfrac{1}{2} \circlearrowright + \tfrac{1}{2} <^{\lambda}_{\mu}\right),$$

or (*vide* Fig. 3, page 246)

$$x = z \cot\left(\tfrac{1}{2} \circlearrowright + <^{\lambda}_{i}\right),$$

and

$$x = z \cot\left(\tfrac{1}{2} \circlearrowright + <^{\lambda}_{k}\right).$$

Introducing j with an indeterminate co-efficient, we have two planes perpendicular to λ and μ, cutting the surface in circular sections. And this result we should be led to expect from the fact that the *paraboloid* is a limiting case between the *ellipsoid* and the *hyperboloid*.

This one example is sufficient to show that, with the property of self-conjugation, the general equation of the second degree loses that simplicity of expression which makes quaternions so singularly applicable to the central quadrics. The very form of the self-conjugate function exhibits some of the fundamental properties of these central surfaces. The *rectangular* transformation depends on the principal axes; the *cyclic*, on the relations of the cyclic normals; while the *focal* shows at once the properties of the focal lines of the asymptotic cone. Other transformations could doubtless be discovered, embodying other well-known properties of these remarkable surfaces.

CAMBRIDGE, May 28, 1877.

XIX.

ON THE SYNONYMY OF SOME SPECIES OF UREDINEÆ.

By W. G. Farlow.

Presented Feb. 13, 1878.

In the "Bulletin of the Bussey Institution," Vol. II. No. 20, reference was made to two species of *Uromyces;* one found on *Spartina stricta*, and the other on *Brizopyrum spicatum*. In the paper above named, the occurrence of the two species in America was considered in detail, but their identity with European forms was barely noticed. The species on *Spartina*, which was referred to *Uromyces Junci*, Schw., was said to agree with the *Puccinia Junci* of Chevalier's "Flore de Paris." The latter species seems to be the *Uromyces Junci* of Tulasne mentioned in "Annales des Sciences Naturelles" 4 sér., tom. 2, which is founded on *Puccinia Junci*, Desm. The date of the publication of Schweinitz's *Puccinia Junci* is 1831, and the species of Desmazières appeared some years later. The species on *Bryzopyrum* appears to be the same form as that referred by Schroeter in "Beiträge zur Biologie der Pflanzen," Vol. I. pt. 3, p. 7, to *Uromyces Dactylidis*, Otth, in which he includes *Uromyces graminum*, Cooke.

In the same article in the Bulletin, *Peronospora obliqua*, Cooke, growing on *Rumex crispus*, was referred to *Ramularia macrospora*, Fres. There has lately appeared, in a recent number of the "Naturaliste Canadien," a list of fungi found near Quebec, determined by Von Thümen. Amongst other species is quoted *Ramularia obovata*, Fuckel on *Rumex*. This probably refers to the same fungus as that mentioned in the "Bussey Bulletin." If, as is probable, *R. macrospora*, Fres., and *R. obovata*, Fuckel, should be united, the name of Fresenius claims priority.

An interesting *Puccinia* and accompanying *Aecidium* found by Mr. Cleveland on *Malvastrum marrubioides* at San Diego, Cal., was referred to *Puccinia Malvacearum*, Mont., with considerable doubt, as a variety differing from the type in the shape and color of the spores.

The fungus is none the less interesting when compared with *Puccinia lobata*, B. & C., collected by Wright on *Sida lepiota* in Texas. The spots are of the same color as in the last-named species, but the spores are decidedly smaller in *P. lobata*, and more obtuse. What we have called *P. Malvacearum*, var. *Malvastri*, is exactly intermediate between *P. Malvacearum*, on the one hand, and *P. lobata*, on the other, agreeing in the size of the spores with the former and in color with those of the latter. The spores are not so acute as in *P. Malvacearum*, nor so obtuse as in *P. lobata*. The connection between *P. Malvacearum* and *P. lobata*, which the form collected by Mr. Cleveland seems to indicate, is, to say the least, worth consideration.

XX.

METASOMATIC DEVELOPMENT OF THE COPPER-BEARING ROCKS OF LAKE SUPERIOR.

By Raphael Pumpelly.

Presented Jan. 9, 1878.

Lake Superior is divided into two structurally distinct basins by the long peninsula of Keweenaw Point. The western basin lying between this point and the north-western shore of the lake is a geoclinal trough, bordered, and perhaps bottomed, by an immense development of volcanic rocks, in the form of great beds or flows. The south-eastern lip of this trough consists of these beds associated with conglomerates and sandstones, both of which consist essentially of porphyry detritus. These rocks, for which Major Brooks has proposed the name of the Keweenaw series, have a minimum actual thickness of more than two miles, and a linear extent, in Michigan and Wisconsin alone, of between two and three hundred miles. As a rule, they preserve a very marked uniformity of character throughout this area. Messrs. Foster and Whitney and Owen considered them as eruptions of the Potsdam epoch, basing their opinion chiefly on the external resemblance of the sandstones interbedded with the eruptive rocks to the closely adjacent Potsdam sandstone, and on a supposed conformability between the two series.

The State Survey, and the private surveys of Major Brooks and myself in Michigan, and later, in Wisconsin, those of Professor Irving, of the Wisconsin State Geological Survey, have discovered abundant evidence of non-conformability, and of the greater age of the Keweenaw series. This is, in fact, much more nearly conformable to the underlying highly tilted Huronian schists.*

They are thus the product of the earliest eruption of basaltic rocks to which a proximately definite age can be assigned. They were preceded by very extensive eruptions of acid rocks, especially porphyries.*

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* Both these eruptive rocks and the sandstones have been referred to the Triassic period by C. T. Jackson, Jules Marcou, and R. Bell.

These basaltic rocks have been subjected to a wide-reaching alteration, which has produced marked changes in the internal condition of the beds, and has filled the fissures with a rich variety of minerals, whose constituents were derived from the products of this alteration.

In a previous article,† I sketched the relative ages of the minerals, especially those occurring in the veins.

In the present paper, I shall try to trace the changes that have taken place in the interior of the rock masses, in places where the only ingress and egress was through the capillary cracks formed by the cleavage and mutual boundary planes of the crystalline constituents.

The results at which I arrive are based on the study of several hundred thin sections, made by myself in order that I might study them before covering. The method followed was in general as follows: —

I. Examination of freshly broken surface under the microscope.

II. Measurement of the angles of crystals, generally cleavage angles, directly under the microscope, or with larger individuals on a reflecting goniometer.

III. Examination in detail of uncovered thin sections, trying hardness with the point of a delicate needle (which was also magnetized to detect magnetite) under any desired power up to a 4-10 objective.

IV. Etching the uncovered section, first with acetic acid to detect carbonates, then with muriatic acid.

V. Study of the covered section under the microscope, in transmitted and reflected light, and in polarized light.

VI. In many instances, qualitative chemical determinations, by my assistant, Mr. B. T. Putnam, were both necessary and decisive.

VII. A small series of analyses was made for me, by Mr. George W. Hawes and Mr. R. W. Woodward, in the laboratory of the Sheffield Scientific School.

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* For recent discussions of the relative ages of the Silurian sandstone, the copper-bearing series, and the Huronian, the reader is referred to articles by Brooks and Pumpelly (Amer. Jour. Sci., vol. iii., 1874, p. 428); R. Irving (Amer. Jour. Sci., vol. viii., 1874, p. 46); T. B. Brooks (Amer. Jour. Sci., vol. xi., 1875, p. 210); and Reports of the Geological Survey of Canada.

† Paragenesis and Derivation of Copper and its Associates on L. Superior, Amer. Jour. of Sci., 3d ser., ii., 1871, p. 188; also, Geol. Survey of Mich., vol. i., part ii.

VIII. Whenever manageable crystals of feldspar could be got, they were cut parallel to the plane of easiest cleavage, and studied by Des Cloiseaux's method. A modification of Des Cloiseaux's method was used, to aid in determining the feldspar in thin sections of rock cut at random.

In the Report of the Geological Survey of Michigan, after a study based on the older lithological methods, I distributed the different varieties of these rocks under three heads; viz., coarse-grained and fine-grained melaphyres, and melaphyre porphyry. In the same volume, my assistant, the late Mr. Marvine, mistaking the augite for hornblende, described the coarser varieties as diorites, which was, at that time, a very common mistake.

In writing this paper, I had intended to retain my original classification; but, at the last moment, I received Rosenbusch's "Mikroscopische Physiographie der massigen Gesteine," in which the *plagioclase-augite* rocks are distributed in the following manner: —

I. Pre-Tertiary.

1. *Granular.*
    - *a.* Plagioclase-augite = Diabase.
    - *b.* Plagioclase-augite-chrysolite = Chrysolitic diabase.
2. *Porphyritic;* containing more or less unindividualized base.
    - *a.* Plagioclase-augite = Diabase porphyrite.
    - *b.* Plagioclase-augite-chrysolite = Melaphyre.

II. Tertiary and Post-Tertiary.

Granular or *Porphyritic.*
- *a.* Without chrysolite = Augite-andesite.
- *b.* With chrysolite = Basalt.

In thus designating as melaphyre only such rocks as are the older equivalents of those feldspar-augite basalts in which the original fluid magma is more or less represented by unindividualized base, it seems to me that Rosenbusch has simplified the troublesome problem involved in assigning to their proper places a large number of partially allied rocks.

The eruptive rocks of the Keweenaw series, or at least all those treated of in this paper, fall readily under the two heads of Diabase and Melaphyre.

The greater number of the beds are fine-grained diabase; coarse-grained diabase is much rarer; and intermediate between the two, as regards frequency of occurrence, is the melaphyre.

The fine-grained diabases are dark-brown, or gray, and green, with subconchoidal to irregular fracture, generally tough, but not hard, on account of the relative abundance of chlorite. Under the microscope, they are found to consist exclusively of triclinic feldspar, — sometimes associated with orthoclase, — and of augite and magnetite, and the alteration products of these minerals. They contain no chrysolite, nor any remnants of the fluid magma in the interstices between the constituents. The feldspar crystallized first, and the interstices were occupied by the augite and magnetite. The feldspar crystals are generally in square prisms, from a lengthening of the zone $O : i\breve{i}$.

The feldspar appears, from measurements, of the angle between the principal sections in alternating hemitropic bands by Des Cloiseaux's method, to be oligoclase in some instances, in some anorthite, and in others labradorite.

In some of the fine-grained varieties which occur more rarely, there is very little augite, and the rock has a light-gray color; this is notably the case with a great thickness of rocks on the Hungarian location, and to a much less degree with beds 45 and 66 of the Eagle River Section.

The coarse-grained varieties, with marked differences of external appearance as regards color and grain, have the same characters under the microscope as the finer grained, except that they contain much more numerous crystals of apatite, and that they often differ in regard to the secondary products.

The melaphyres are sharply marked. The constituents are plagioclase, augite, magnetite, chrysolite, and remnants of the fluid magma, occupying, in more or less small quantity, the interstices between the constituents. The chrysolite is generally more or less altered, and the remnants of the fluid magma are only rarely preserved in the form of glass base, for instance, in bed 90, Eagle River Section; it is almost always altered to a green chloritic substance. Beds Nos. 108 — "the Greenstone," — 90, and 64, all in the Eagle River Section, are types of the melaphyre, and to it belong, generally, what are locally known as mottled traps.

In the melaphyre, the feldspar appears to be generally anorthite, to judge from the size of the angles between the principal sections in alternating hemitropic bands in sections cut at random in the zone $O : i\bar{i}$. It is in sharply defined, long, narrow crystals, enclosed in the younger augite.

The melaphyre is less subject to the changes that produce the intermediate or amygdaloidal form, and is not so apt to have a true amygdaloid for the top of the bed.

Not a trace of hornblende was observed in any of these rocks; it occurs with orthoclase in some related coarser-grained rocks of the Keweenaw series near the Wisconsin State line.

The almost general acceptance of Tschermak's theory, which breaks down the boundaries between the triclinic feldspars, has rendered the determination of these a matter of wholly secondary importance in diagnosis.

But the feldspar plays such an important part in the series of changes that have taken place in these rocks, that it seemed desirable to determine its position in the plagioclase series. This is no easy task. Except in a few instances, chemical analysis must necessarily fail, because the rocks have generally changed in character, — either the feldspar or augite being more or less altered; the composition of the augite is an unknown quantity; and isolated crystals of both minerals are too small to be got out clean for analysis. The analyses I had made were of little use in determining the feldspar. The entering angle on the basal cleavage seemed to offer a ready means of determination by direct measurement. I therefore made a long series of careful measurements with a reflecting goniometer, with very unsatisfactory results; for the cleavage surface in one or both sets of hemitropic individuals is apt to be either curved or crossed by innumerable smaller striae, each giving an image, so that the result lies between extremes often from thirty minutes to one degree apart, or more than the difference between albite, oligoclase, and labradorite, and in addition to this, different authorities give different measurements for this angle. Still I give the results in the general table.

Des Cloiseaux's beautiful method is much more promising. This consists in cutting very thin sections parallel to the basal cleavage, and measuring, between crossed nicols, the angle between the edge $O \mid i\breve{\imath}$. and the principal section — maximum extinction of light, — or between the maximums extinction in the alternating hemitropic bands to guard against errors that might arise from obliquity in the section. Des Cloiseaux determined this double angle, from many measurements, to be: —

Oligoclase	0° — 2°
Andesine	3° — 4°
Albite	7° 40′ — 9° 40′
Labradorite	10° 34′ — 13° 56′
Anorthite	40° — 80°
Microcline	30° 54′

Aside from the fact that it is very rare to find a crystal in these rocks large enough to manage in cutting: *i. e.*, not less than $\frac{1}{20}$ inch square, there was the possibility that these might be different from the dominant feldspar of the rock. The results obtained by this method are given in the table.

It occurred to me that this method might be modified by determining the range of this angle in the zone *O : iī*, and measuring the angle between extinctions in several individuals in a section, always taking care that the angles should be equal or nearly so in both bands.

The following is a proximate determination of the range of the angle included between the principal sections of the alternating bands in the zone *O : iī* for the different feldspars. These values are based on the position of the optic-axial plane and of the bisectrix, as established by Des Cloiseaux.

	Minimum.	Sections cut parallel to *O*.	Maximum in the zone *O i ī*.	Between line of extinction and edge *O*|*i ī* sections parallel *i ī*.
Orthoclase	0°	0°	0°	4° — 5°
Albite	0°	7° 30′ — 10°	32° — 34°	15° — 20°
Oligoclase and Andesine . .	0°	0° — 4°	36°	0° — 7°
Labradorite	0°	10° — 14°	62°	20° — 27°
Anorthite	0°	Sections in *i ī*. 20° — 26°	Sections in *O*. 40° — 80°	33° — 46°

From this table, the angle in question appears to stand in an inverse relation to the acidity. By measuring a considerable number of twins in a thin section, taking care to select only such as form equal angles on the opposite sides of the cross hair, it is often possible to distinguish anorthite and labradorite, but it is scarcely practicable to discriminate between albite and oligoclase by this method alone.

In giving the result of these measurements in the general table, I have given the amount of the double angle, and have put down only those measurements in which the angles in the two members of a twin were even, or very nearly so. Where the angle was $\frac{1}{5}$ or $\frac{1}{6}$ greater in one member than in the other, the result is given in italics.

This is the only method that could be used in these rocks, which are generally too much altered to admit of a calculation of the mineral constitution from chemical analyses; and the feldspars are apt to be so much affected that the tests for fusibility, and by etching with acid, are useless. But the smallest unaltered portions of a crystal answer readily to the optical tests, especially if the section is cut thin. By using a circular polarizing quartz plate 3.75 millimetres thick, as recommended by Klein and Rosenbusch, inserted just above the objective, and by turning the analyzer till a violet tint is obtained, the readings can be made with much greater precision than by observing the maximum of darkness. In the presence of the quartz plate, all the double-refracting portions of the section appear in different colors, excepting where a principal section coincides with a nicol plane; crystals so placed will be violet, and a very slight disturbance will change the color to blue or red.

The chief defect in this method is in the fact that, used alone, it detects only the feldspar that gives the highest angle, while there may be two kinds present.

The coarse-grained varieties differ very markedly in external appearance. There are several varieties, some of which are common to the most distant portions of the copper district; they are marked by difference in grain, texture, and color, and in the nature of their secondary constituents. This last difference is apparently largely due to variation in the feldspars. The coarser forms have more or less orthoclase, and in some this may predominate. The triclinic feldspar varies in character from oligoclase to anorthite.

The pyroxene is to all appearances the same augitic variety in all, and in all instances it crystallized after the feldspars and occupies the interstices between these. Only in rare instances — as in bed No. 95, Eagle River Section — a well-defined cleavage seems to mark the pyroxene as diallage.

The specific gravity varies from 2.89 to 3.03; the colors from dark-brownish black, with resinous lustre, as in bed No. 95, to a very light gray, as in bed No. 107. Between these extremes, there are varieties of many shades of green and brown, or red. Indeed, specimens of the eight or ten varieties differ so widely in external appearance, that it is difficult to believe that they are closely allied to each other, and to the finer-textured rocks, until we see the similarity of constitution and texture as regards primary constituents, and also their similar mode of occurrence, and the fact that they are interbedded with the finer-grained diabase. Such of these coarser forms as exhibit interesting phases of alteration will be described. I will remark here that Mr. Marvine, in his otherwise very excellent description of the beds of the Eagle River Section (Geol. of Mich., vol. i. part 2, pp. 119–

140), wrongly designated these coarser rocks as diorites. I refer the reader to these very faithful descriptions, by Mr. Marvine, of the external characteristics of these rocks, adding only the caution that, wherever he mentions hornblende, it should read augite.

As it is the object of this paper to describe the internal changes that have taken place in these rocks, rather than a description of the rocks themselves, the sequence in which they are described is that which is best adapted to the immediate purpose.

Earliest changes in Melaphyre.

"*The Greenstone.*" The rock, commonly known in the copper district as "the greenstone," is the best type of this subdivision. Forming a bed 400–500 feet thick, dipping northerly, its outcrop consists of a continuous series of ridges and nearly vertical escarpments, often several hundred feet high. These extend from the Allouez mine to Point Keweenaw, and form both the most salient topographical feature of the peninsula, and a well-defined horizon, to which the geologist and the miner refer their measurements. The rock has suffered only to the extent of a partial alteration of one of its constituents, — the olivine.

It is dark-green, greenish-black, finely crystalline, very compact, hard, and brittle, and breaks with an uneven to semi-conchoidal fracture. Porphyritic crystals, apparently of orthoclase, from ⅓ inch in length down, occur here and there, — one or two on the surface of a specimen. They are generally single individuals; but sometimes twinned after the Carlsbad law, as is shown by the oppositely inclined basal cleavage. The powder of the rock yields to the magnet a beard of magnetite. The specific gravity is 2.92–2.95. It is an important characteristic of this rock, that its freshly fractured surface is mainly occupied by spots ⅓ to ¾ inch in diameter, each of which reflects the light with a satin-like sheen. The reflection is not carried to the eye from all the spots at once: it is generally necessary to change the position of the specimen many times to observe the different reflections. Aside from this sheen, there is nothing, either in difference of color or texture, visible to the naked eye, to betray the presence of these spots, which might be called lustre-mottlings.

To the naked eye, this phenomenon suggests, at once, interrupted cleavage of large individuals of one of the constituents, as the cause; but, under a strong hand-glass, these reflecting surfaces show the same granular texture and character as the rest of the rock; and it is only when examined under the microscope, with an objective of low power and in polarized light, that the appearance to the unaided eye is corroborated. We here find the cause in the fact that each spot is the cross-fracture or cleavage of a crystal of pyroxene, which in crystallizing has enclosed hundreds of feldspar crystals.

The weathered surface is rusty gray, scarcely $\frac{1}{50}$ inch thick; but it is covered with knobs which are due to the more rapid destruction of the materials between the pyroxene individuals.

Examining thin sections under the microscope, we find the constituents to be *plagioclase*, *pyroxene*, *olivine*, and its alteration product, as well as magnetite, and an unindividualized substance, both fresh and altered, occupying interstices.

In thin sections, the plagioclase is seen to exist in very sharply defined and fresh, thin, tabular crystals, .001 to .002 inch thick, and .01 inch, and less, long. It contains scattering interpositions of an opaque black substance, and minute brown particles, which may be, or have been, glass.

The crystals of plagioclase have predetermined the contours of all the other constituents, except the olivine, which crystallized first.

The predominating feldspar is anorthite, as determined by the angle between the principal sections in adjoining bands in the zone $O : i\bar{\imath}$. Scattering large crystals, which happen in the sections to be cut parallel to $i.\bar{\imath}$., have their principal sections at an angle of 23° with the edge $O : i\bar{\imath}$, which would indicate albite or labradorite.

The augite is very fresh and transparent, almost colorless in the thin section, but with a tendency to purple-gray. An imperfect cleavage is indicated by somewhat irregular parallel fractures. It fills the interstices between the closely packed individuals of feldspar in such a manner that a single pyroxenic crystal encloses many hundreds of these, while its crystalline integrity is shown by the uniform color in polarized light, and by the arrangement of the cleavage cracks throughout the area of the augite individual.

It is a remarkable fact that, while these large individuals of pyroxene contain thousands of feldspar crystals, they enclose only very few of olivine or of magnetite. These minerals, together with the unindividualized substance, are crowded into the spaces between the pyroxenes. In this intermediate space, which surrounds the pyroxene individuals with a continuous network, we find, also, a few small pyroxenes, just as isolated grains of olivine occur in the pyroxene areas.

A careful examination of this occurrence will, I think, convince the observer that, at the time the pyroxene crystallized, both the olivine and the feldspar crystals, and apparently the magnetite, were already individualized; for, where we find any of these in contact with the pyroxene, we see that the latter has adapted itself to the already defined contours of the others. While the pyroxene enclosed the feldspar crystals with ease, it crowded the other constituents almost wholly into the surrounding spaces; a process which was facilitated by the presence of the then fluid, unindividualized substance.

The pyroxene contains, also, brown interpositions, similar to those in the feldspar, and some opacite; and, where it occurs with olivine, it often surrounds the grains of this mineral.

The magnetite bodies are of irregular shape, moulding themselves sharply around the contours of the feldspar and olivine. Their sharply defined outlines are black and fresh.

The olivine is abundant in integral and aggregated grains, and, very rarely, in crystals with recognizable though rounded contours. The individuals are .0008 to .005 inch in diameter, and are readily distinguishable from the augite, between crossed nicols, by the difference of colors, and in the crystals, by the parallelism of the principal section with the longer sides. Where pains have been taken to give a tolerably good polish to the surfaces of the thin section, the characteristic finely pitted surface of the olivine distinguishes it, even in ordinary light, from the augite, which takes a more perfect surface.

There are few grains of olivine in this rock that are not more or less altered to a very pale green substance, sometimes tinged with brown. Under a high power, the olivine is seen to be traversed by intercommunicating canals .0002 to .00005 inch thick, filled with a clear, faint yellowish green to greenish-blue substance. From the sides of these channels, jagged points of the same substance penetrate the fresh olivine. In this manner, larger or smaller parts of the grains have been changed to a feebly double-refracting substance which gives an aggregate polarization due to the arrangement of the minute individuals of the alteration product, which are sometimes felted, at others, parallel fibrous. This product is dichroitic; pale green when the fibrous structure is parallel to the shorter diagonal of the polarizer, and pale orange when parallel to the longer diagonal. On the uncovered sections, this alteration product was found to be very soft under the needle.

Apparently, not more than twenty to thirty per cent. of the olivine is altered, which is very remarkable in a rock of such great age, considering the fact, which is emphasized by Zirkel,* that the olivine is subject to complete alteration, even when its neighbors remain wholly intact.

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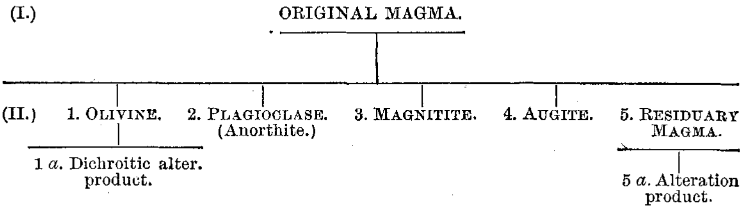
* Mikrosc. Besch. d. Mineralien u. Gesteine, p. 217.

Where the interstices between the constituent crystalline minerals are not occupied by augite, they are filled with a transparent substance, in places colorless, in others of faintest green, almost colorless, only just distinguishable from the colorless feldspar in ordinary transmitted light. It often penetrates the feldspar crystals in cross fractures. Between crossed nicols it is black, sprinkled with minute blue-gray clouds, — an aggregate polarization due, probably, to either an exceedingly minute radiating fibrous, or a granular, structure. Under a $\frac{1}{16}$ inch objective (Hartnack's, No. 10), portions of the black seem to remain dark on revolving between crossed nicols. The substance is at the most only very feebly double refracting. Unfortunately I was able to find with a low power, and to try with the needle on an uncovered section, only those portions which were faintly green, and these were soft. Still, I am forced to believe that we have here to do with remnants of glass base, which is altered in the faint green portions to a chloritic or serpentine substance. Its whole mode of occurrence in this very fresh rock shows conclusively that it is not an alteration product of any of the constituent crystalline minerals.

The relative ages of the constituents of this rock appear to be well defined as follows: —


(I.)
ORIGINAL MAGMA.
(II.)
1. OLIVINE.
2. PLAGIOCLASE. (Anorthite.)
3. MAGNITITE.
4. AUGITE.
5. RESIDUARY MAGMA.
1 *a*. Dichroitic alter. product.
5 *a*. Alteration product.

The residuary magma (5) must have been the last to solidify; to its presence was due the internal mobility of the mass, which rendered it possible for the augite to crystallize in larger individuals, and, in doing so, to crowd from its centres the olivine and magnetite individuals.

This residuary base, probably, differs in chemical constitution from the original magma, since it is only the residue of this after the removal of the ingredients forming the crystalline constituents.

Highly altered Melaphyre. — While the type represented by "the greenstone" is one of the most constantly recurring varieties throughout the copper district, it is very rarely in as unaltered a condition, and it is usually finer grained and almost aphanitic in texture. Where it is least altered, the fresh fracture has less lustre, but the characteristic lustre-mottlings are present to the same extent, though in smaller spots.

An instance of a highly altered form of this variety occurs at the shaft on Mabb's vein near Houghton. The specimens are from the immediate neighborhood of the fissure vein; and the one on which the following observations were made represents the contact: —

It is about 5 × 4 inches, and carries at one end $\frac{3}{4}$ of an inch thickness of the vein, in the form of a parallel arrangement of calcite-quartz seams, separated by thin chlorite layers.

In close connection with this altered form there is an intermediate stage which has an uneven fracture, and is distinguishable from the "greenstone" only by a somewhat finer texture, and the smaller size of the lustre-mottlings. The magnet extracts a little magnetite from the powder. In thin sections, the characteristic structure to which the lustre-mottlings are due appears at once between crossed nicols, and the same crowding of olivine and magnetite individuals into the space surrounding the large pyroxene areas. Both the feldspar and pyroxene appear to have suffered but little from alteration; but the chrysolite individuals are all altered, and the unindividualized substance is changed to a green non-dichroitic substance.

The more altered form has an even to semi-conchoidal fracture. The freshly broken surface is formed of nearly round, green spots, about $\frac{1}{4}$ inch in diameter, in a connected network of brown.

Fragments of the rock effervesce strongly in muriatic acid. Its powder does not yield magnetite.

The structure that in the less affected rock causes simply lustre-mottling is, in this altered form, marked by a decided color-mottling.

In thin sections, the plagioclase crystals show still, to a great extent, the twin striation in polarized light.

The olivine individuals are changed to a slightly yellowish green substance with brown stained cracks, and with the outer portion also stained brown. They are very soft under the needle.

Examining these with a strong glass, on a fresh fracture of the rock, they show a well-marked cleavage in one direction, — an irregular fracture in all others, — and have a dark red color with somewhat wax-like lustre. They blacken before the blowpipe, and fuse with difficulty on the edges of very small fragments.

In the thin sections, many of the individuals are so stained, in part or throughout, by oxide of iron as to be wholly opaque; but by etching with muriatic acid, the discoloration is removed. The staining is, therefore, probably due to a deposition of the oxide in the cleavage planes.

Under the microscope, the cleavage is very evident in many individuals, but it is more marked in the less altered rock. In those instances where the cleavage is well defined, an examination with one nicol show decided absorption, — the substance being almost colorless when the cleavage lines coincide with the longer diagonal of the nicol, and delicate green when perpendicular to this.

Between crossed nicols, darkness occurs when the cleavage lines coincide with a nicol-plane; and some individuals which show no cleavage lines remain dark during a revolution.

This should seem to indicate that the substance is uniaxial.

Many of these grains, especially in the more altered rock, differ from the above in that they show an aggregate polarization due to a confused fibrous or felted structure.

The pyroxene in the more altered rock is apparently wholly changed to an impellucid white substance, which, between crossed nicols under a high power, shows a brilliantly variegated aggregate polarization, an irregular damask arrangement of bright-colored lines, forming concentric circles or parallel wave-like lines. This substance is not confined wholly to the pyroxene: it occurs also in the form of irregular veins which have a rough tendency to parallelism, and no defined walls but which extend through the feldspar crystals as well as through the other constituents.

One of the specimens of this rock has at one end fragments of the Mabb's vein in the shape of five or six lenticular layers of calcite, with quartz $\frac{1}{16}$ — $\frac{1}{8}$ inch thick, separated by still thinner layers of the rock. A close examination of the specimen near this portion reveals the presence of very minute veinlets of calcite parallel to these larger ones, while a still closer observation of the weathered surface shows that still finer parallel veinlets occur throughout the specimen.

Under the microscope it became evident that some of the veinlets contained quartz as well as the calcite.

On etching a thin section with acetic acid under the microscope, the removal of the calcite was observed, and it became evident that the irregular veins of impellucid white substance (which I mentioned as extending beyond the pyroxene crystals) followed these veinlets, the calcite forming the median line of the impellucid veins. The acetic acid removed the calcite through to the under side of the section, but the impellucid substance remained; nor was it appreciably changed by treatment during twenty-four hours with cold muriatic acid, which removed considerable iron from the altered chrysolite.

On etching thin sections with acetic acid, during a period of forty-eight hours, it was found that not only was calcite removed from the impellucid veinlets, but also extensively from the representatives of the pyroxene. The action of the acid revealed the presence of numerous cracks in the latter, which were filled with a colorless, more or less transparent, and double-refracting substance not acted upon by the acid. On examining the etched section by reflected light, and afterwards in polarized light, the former positions of the pyroxene individuals were found to be occupied by a thoroughly honeycombed mass; the crystalline substance that had been formed in the cracks now forming the partitions between cells, which contained an apparently confused granular mass of white translucent substance. On the thinner edges of the section the cells were often entirely empty, perhaps owing to mechanical action in washing after etching.

The plagioclase remained, as before the treatment with acid, apparently fresh, except in immediate contact with the calcite veinlets, and also where former interpositions of glass base had been changed to a greenish double-refracting substance.

That the ferric oxide which stains the pseudomorphs after chrysolite, and also occupies capillary cracks in other constituents, was formed from the iron in the chrysolite, seems to be apparent from the fact that it exists in the less altered form of the rock where the pyroxene is perfectly fresh.

That this staining is older than the calcite appears from the fact that the calcite veinlets cut the stained cracks, and are never themselves discolored. But the magnetite appears to have been oxidized at a later date without spreading a discolorization.

The chrysolite was the first to suffer, for we find it wholly changed in the less altered form of the rock. The change, possibly, consisted in a hydration, accompanied by separation of iron.

There occur seams $\frac{1}{100}$ inch thick of a green, non-dichroitic chlorite, which shows lamellar aggregate polarization. These seams are traversed by the calcite veinlets, and are, therefore, older.

The quartz, both in the larger vein fragment and in the veinlets, seems to be younger than the calcite, though the evidence is not as positive as could be wished. In a section cut from the vein fragment, I observed a quartz crystal cut nearly perpendicular to the principal axis, and surrounded by calcite: it seems that the growth of the quartz was interfered with; for its development is marked by successive concentric layers, and while the central ones are complete there was evidently not room for the outer layers on one side, owing to the proximity of the face of a calcite crystal against which they end abruptly.

In some instances, the quartz shows a comby structure, and the interstices between the opposing ends — the median line — are occupied by a little green, chloritic substance.

The process of growth of the small veins near the fissure vein is well defined under the microscope. Throughout the body of the specimen, the calcite is confined to the pseudomorphs after pyroxene, and to the minute crooked veinlets. But near the end bearing the vein fragment, a greater regularity in the form of the minute veins is apparent, and is evidently due to the lesser resistance offered in the direction of the fissure. The veinlets unite, to form wider veins; and in doing this the tributary always enters between the larger veinlet and the rock, and maintains this position. Each member of the compound vein is marked by its independent crystallization, and by the narrow borders of impellucid substance, and the whole appearance is best comparable to the map of a glacier which has received several tributaries, each marked by its lateral moraines.

The greater facility for drainage indicated in this structure has had. for effect, near the vein, the removal of the calcite from the pseudomorphs after pyroxene, which appear now to be filled with chlorite and quartz.

Minute particles of native copper occur in these veinlets. An examination of fragments containing the copper flakes, under the microscope in reflected light, showed the metal to be younger than the quartz, for it adapts itself to the contours and cracks of the latter.

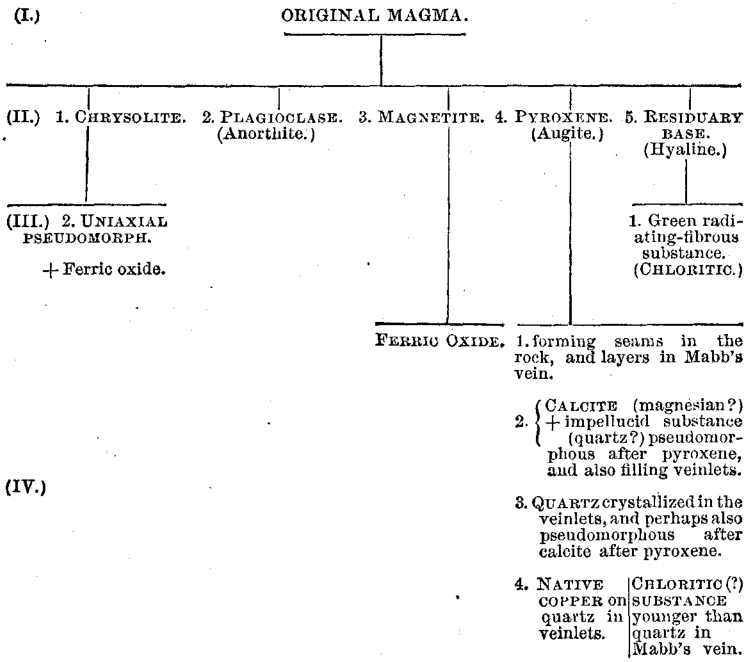
Throughout all this change the plagioclase appears to have suffered little, except in the immediate contact with the fissure vein, where it has lost in most individuals the evidence of twinning in polarized light. It is also, perhaps, affected in the immediate contact of its crystals with the veinlets, but it is generally so fresh that it cannot have contributed appreciably to the changes that the rock has undergone.

The first change — that affecting the chrysolite and the residuary base — may have been simply of the nature of a hydration of these, accompanied, in the former, by a separation and oxidation of the iron. This represents the condition of the "greenstone," in which this change is partially accomplished, and of the similar "mottled traps" throughout the copper district, in which the alteration of the chrysolite and residuary base is, probably, generally complete.

The next change has affected the magnetite, and still more the pyroxene, the destruction of which has given birth to, at least, three products; namely, *chlorite*, *calcite*, and *quartz*. The relative ages of these, as expressed in the accompanying paragenetic table, are as observed where they are all represented in the veinlets. In the breaking up of the pyroxene combination, the iron, alumina, and magnesia, and part of the silica were removed, and went to form chlorite; the lime, combined with the foreign carbonic acid, — possibly, also, with some magnesia, — remained as a pseudomorph: there is less positive evidence that the substance occupying cracks in these pseudomorphs (and which is not affected by a twenty-four hours' treatment with cold muriatic acid) is quartz, and that it is pseudomorphous after calcite.

The only remark to be made with regard to the native copper is that it was observed only in the veinlets near the fissure vein, and as being younger than the quartz.

The paragenetic scheme for this rock should seem to be as follows: —


(I.) ORIGINAL MAGMA.
(II.) 1. CHRYSOLITE.
2. PLAGIOCLASE. (Anorthite.)
3. MAGNETITE.
4. PYROXENE. (Augite.)
5. RESIDUARY BASE. (Hyaline.)
(III.) 2. UNIAXIAL PSEUDOMORPH.
+ Ferric oxide.
1. Green radiating-fibrous substance. (CHLORITIC.)
FERRIC OXIDE.
1. forming seams in the rock, and layers in Mabb's vein.
2. CALCITE (magnesian?) + impellucid substance (quartz?) pseudomorphous after pyroxene, and also filling veinlets.
(IV.)
3. QUARTZ crystallized in the veinlets, and perhaps also pseudomorphous after calcite after pyroxene.
4. NATIVE COPPER on quartz in veinlets.
CHLORITIC (?) SUBSTANCE younger than quartz in Mabb's vein.

CHANGES IN THE LOWER PARTS OF THE BEDS.

There is no variety of these rocks that has not undergone a greater or less change. The least altered instance is that which forms the "Greenstone range," in which the remnants of magma-base and part of the chrysolite have been changed, while a considerable portion of the chrysolite, and all of the plagioclase, pyroxene, and magnetite, remain apparently quite intact. The further change which this variety has undergone under favorable circumstances, namely, the close proximity of a vein, has been already described. I propose now to consider the evidence, and, so far as I have been able to trace it, the progress, of the alteration which has taken place in the body of the rock, especially in the diabase proper, away from the influence of veins, and where the influx into, and drainage from, a given point, could have taken place only through capillary cracks and the cleavage planes of the crystals. As I have shown elsewhere, all the beds have very sharply defined hanging walls and foot-walls, while between these limits the rock has, generally, three subdivisions; namely, a lower — least altered zone — below, a pseudoamygdaloid in the middle, and amygdaloid at the top. The upper one, and sometimes the middle one, are, in places, wanting. The relative thickness of these subdivisions is very variable, not only in different beds, but also in different parts of the same bed; and there is no plane of separation between them. This classification is based on the presence and character of the amygdules in the middle and upper division, and their infrequent occurrence in the lower.

The amygdules in the middle portion, and the more isolated ones in the lower, as a general thing, occupy the place of former constituents; while those of the amygdaloids proper have, to a great extent, been formed in pre-existing cavities of more or less regular form.

The changes that have taken place in the middle and lower portions of any bed are such as tend to produce a pseudo-amygdaloid. The first and ever-present stage of alteration is caused by the change of the residuary magma-base which fills the interstices between the crystalline constituents, and in places penetrates into, or is enclosed in, the interior of these. The physical and chemical character of this seems to have predisposed it to an easy change. It is now, as a rule, when seen in thin sections, a darker or lighter olive-green substance, and very soft under the needle (hardness not over 2.5). In polarized light it exhibits a fibrous aggregate polarization, and shows well its structure, which is short fibrous, converging towards the centre. The central portion shows sometimes little or no double refraction, but more generally it is filled with very minute polarizing spheres formed of radiating fibres. With one nicol, this substance shows only absorption for intensity. The contours are generally sharply defined by the feldspar and pyroxene crystals, and the result is usually a more or less wedge-shaped form.

The next step has been the change of the chrysolite. In the so-called greenstone, this has been only partial; but generally in the chrysolite-bearing beds it is complete. The result in thin sections is a faintly green substance, soft under the needle, and surrounded, within the original contours of the crystal, by a more or less opaque deposit of iron oxide, which also traversed it in fissures. The green substance shows by a well-defined cleavage in one direction that it is in thin laminæ. Between crossed nicols, these laminæ have an appearance of twin structure, polarizing the light in alternate lines of brilliant red and green. The whole pseudomorph becomes dark when the cleavage is parallel to a nicol plane; and some individuals, probably cut parallel to the cleavage, remain dark through a revolution of the stage. The substance is, therefore, very probably uniaxial. It has very appreciable absorption for intensity, and very feeble for color.

The augite was the next to undergo change. Generally in any thin section of the lower portion of a bed, a considerable proportion of the pyroxene is fresh, either throughout whole individuals or in parts of these.

In thin sections, by ordinary transmitted light, the pseudomorphous product is translucent, faintly light green, with a tinge of brown. Between crossed nicols, in its most characteristic form, it shows irregular lamellar aggregate polarization. It is very soft under the needle, and is traversed by red stained cracks corresponding to the irregular fissures in the parent pyroxene, and it is by these, together with the structure as seen in polarized light, that it is generally best distinguished from the product after residuary magma-base.

The mineral forming these pseudomorphs is very probably the result of a process which has removed lime and some iron, magnesia, and silica from the pyroxene and brought in water, and it is, probably, poor in alumina.

The plagioclase is generally the last constituent that has been altered to any great extent. The most usual product is chloritic. It is very usual to observe very minute particles of a green, apparently structureless substance, suspended in the interior of the feldspar in such a manner as to render the supposition quite possible that they are due to an alteration of enclosed particles of hyaline base. But an actual pseudomorphism of a chlorite after plagioclase is observable on a large scale. In the first stages, small tuft-shaped particles, consisting of laminæ or fibres radiating from a point, occur scattered through the interior of the feldspar, and these may wholly occupy a considerable portion of a crystal, while the rest still shows twin striation in polarized light. In the finished state no trace of the feldspar is visible except the outlines. The pseudomorph then shows an aggregate polarization due to a confusedly felted mass of minute chlorite tufts. The substance is poorly characterized.

The green substances which I have described so unsatisfactorily, as being undoubtedly pseudomorphous after residuary-base, pyroxene, and plagioclase, have in themselves no physical properties, recognizable under the microscope, which are sufficiently *persistent* to invariably characterize them respectively. Often the best means of distinguishing between them is in the internal structure of the aggregate, since this is intimately connected with certain physical characteristics of the original mineral which predetermined the mode of growth and gradual arrangement of the secondary product. In other instances, the presence or absence of cracks stained with iron oxide are characteristic.

Thus the change of the residuary base has resulted in a tendency to form bands parallel to the contour of the area, in a manner that indicates a gradual progress along concentric shells from the circumference inward or the reverse; while in the feldspar the growth appears to have begun without any regularity throughout the cleavage and twinning planes of the crystals. The pseudomorphs, after pyroxene, are almost invariably defined by the red-stained cracks, and slight mixture of brown in the green.

There is another occurrence of chlorite in which the progress of growth is from within outward. Throughout the pseudo-amygdaloid occur grains of chloritic substance, which in places reach a diameter of $\frac{1}{4}$ to $\frac{3}{8}$ inch, with often more or less irregular outlines, often nearly round or oval. These consist of a dark green mineral with $H = 2.5$, which fuses B. B. at 3 — 3.5 to a black magnetic slag.* In different beds, its texture under a hand-glass varies from amorphous to finely scaly. In thin sections, in polarized light, the substance often resembles closely that in the pseudomorphs after plagioclase, except in that it shows evident growth from within outward. There is no defined wall, as of a preexisting cavity, but the chlorite often sends out long arms, which surround or penetrate the adjoining primary constituents. In this manner the chlorite-like pseudomorphs after plagioclase and pyroxene, etc., are sometimes incorporated into these pseudo-amygdules. Very often one of these bodies has a large central area filled with closely packed radiating spheres, surrounded by fragments of a once continuous band with cross-fibrous structure, which evidently once formed the outer limit;

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* For chemical analyses of this chlorite, see beyond under "Bed No. 87."

outside of these fragments is an outer chlorite area, resembling that in the centre, and generally bordered on its outer limit by a narrow cross-fibrous band which adapts itself closely to the primary constituents. The greater number of these bodies seem to have resulted from a gradual change of the primary minerals into chlorite by progress from molecule to molecule. At the first glance, the structure does not seem to confirm this view; for the narrow outer band enclosing a large central filling seems to suggest either, 1st, a pre-existing cavity, on the walls of which the thin outer layer was deposited as the older member, and, within this, the central filling as the younger; or, 2d, the replacement by chlorite of a former secondary mineral, which was attacked at the same time, around its circumference, producing the outer band (shell), and throughout the interior.

Amygdules resulting from both of these processes are abundant in the amygdaloids proper; but they betray their origin in a marked manner, and differ essentially from these pseudo-amygdules.

Whatever the chemical nature of the process resulting in these pseudomorphs, the central area is the oldest member, while the outer band is the younger; and its cross-fibrous structure is only a transitional form destined to be changed to spheres with radiating structure. If we examine the structure of the outer band, we find that its line of contact with the primary minerals, or its axial line, is usually more or less serpentine, and that the cross-fibres, instead of being parallel to each other, are more nearly perpendicular to the axial line of the band, and form closely packed groups, in each of which the fibres radiate from a central point on the axial line, forming minute hemispheres, which bristle towards the interior of the body. The next stage of growth finished the other half of each sphere, and what was a cross-fibred band becomes now undistinguishable from the rest of the central filling, a new band having formed outside of the previous one. In places we find perfectly straight bands, with actually parallel cross fibres, which can hardly be supposed to break up into spheres; and indeed we find that new parallel bands are formed outside of these until the line of attack becomes crooked, when the normal mode of growth is re-established. The remnants of these straight bands are then preserved in the interior of the body.

Bed No. 96. — There are two other instances of pseudomorphs after plagioclase, which occur in the lower zone; viz., prehnite and quartz.

Bed No. 96, lying north of the "greenstone," in the Eagle River Section, is a rather coarse-grained diabase. It is dirty green, filled with small dull-gray feldspar crystals, and presenting on the fresh fracture numerous irregular-shaped, small, chalky-white spots. Examining the broken surface, under a 1½ inch objective, twin striation is well marked on some of the feldspar crystals, while others are too much altered to cleave, and are easily scratched by the needle. Grains of fresh pyroxene are abundant, and the chalky-white spots are found to be hard, and to resemble in structure, on a minute scale, the common prehnite amygdules of the rocks. There is also an abundance of a velvety, dark-green, very soft substance, with conchoidal fracture and resinous lustre, which is often seamed with a brilliant, black, metallic substance, apparently magnetite. This dark-green substance fills spaces between the other constituents, and often surrounds the prehnite-like bodies.

This prehnite-like mineral fuses readily in the flame of an alcohol lamp; it dissolves in heated muriatic acid, leaving pulverulent silica, and, in the filtrate, ammonia produces an abundant precipitate, slightly tinged with brown. There can be little doubt that the mineral is prehnite.

In thin sections, between crossed nicols, the prehnite bodies give the characteristic colors of this mineral, and have often a radiating structure generally starting from one point on the side. Often they consist of aggregated small masses, with the same structure, and too hard to be scratched by the needle.

A considerable portion of the feldspar crystals contain a greater or less quantity of these prehnite tufts, and not rarely the only vestige of the feldspar left is the contour of its crystal. Optical measurements, by Des Cloiseaux's method, on sections cut parallel to *O*, and on random sections in the zone $O : i.\bar{\imath}.$, together show that the feldspar is labradorite. Many of the crystals measured were partly changed to prehnite.

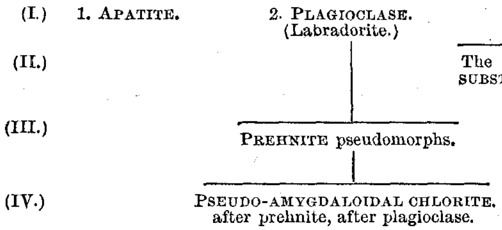
The prehnite is often changed to a more or less soft, white, semi-opaque substance, and is then almost invariably enveloped by a clear, green soft mineral, with radiating structure, which resembles the chloritic substance of the pseudo-amygdules already described. The plagioclase crystals also contain this green substance, and are, in places, represented by it alone.

This green mineral gives a greenish-brown streak, and fuses quietly before the blow-pipe to a black magnetic slag; it dissolves in muriatic acid, leaving a deposit of silica, and, in the filtrate, ammonia precipitates alumina and iron. There can be little doubt that the prehnitization preceded and mediated the formation of the pseudomorphs of this green mineral after feldspar.

The pyroxene has been more or less changed to its characteristic product; and in some instances this contains numerous seams of magnetite, as shown by a magnetized needle in scratching an uncovered section.

The feldspar crystals are in places traversed by long, slender needles of apatite. The pyroxene adapts itself sharply to the contours of the feldspar.

The paragenetic scheme would seem to be —


(I.)
1. APATITE.
2. PLAGIOCLASE. (Labradorite.)
3. PYROXENE.
(II.)
The characteristic GREEN SUBSTANCE and MAGNETITE as pseudomorphs.
(III.)
PREHNITE pseudomorphs.
(IV.)
PSEUDO-AMYGDALOIDAL CHLORITE. after prehnite, after plagioclase.

Bed 800 *feet E. of Hungarian lode.* — The occurrence of the pseudomorphism of prehnite after plagioclase is very beautifully illustrated in an extensively developed series of beds lying east of the Hungarian mine. These beds are remarkable as being almost free from pyroxene.

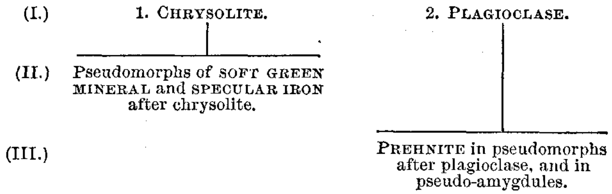
The rock from which I cut my sections is brownish-gray, fine-grained, with even, subconchoidal fracture. Specific gravity = 2.83. The naked eye detects greenish-white, irregular bodies of prehnite, from $\frac{1}{4}$ inch diameter down, and under the loupe the rock is found to be thoroughly impregnated with them. It also contains abundant minute grains of a soft, brown mineral, intimately associated with a brilliant black mineral, with metallic lustre, which is not attracted by the magnet, and is probably specular iron.

In the thin sections, the feldspar is found to have formed fully nine-tenths of the primary constituents. Its sections seem to indicate square, thick crystals instead of thin, tabular forms. A part of that which is still transparent shows no twinning, and some sections were observed in which the principal sections were parallel to the rectangular outlines, and which were probably orthoclase. The few optical measurements made on random sections in the zone $O : i\bar{i}$. did not give any higher results than would be required for oligoclase. No pyroxene was observed; and, while it is possible that some of the soft grains seen with the naked eye are derived from it, the appearances seem to indicate that they are pseudomorphs after chrysolite. They generally consist of a clear, slightly-greenish substance, which is in places amorphous, and in others contains minute fibres which polarize the light very feebly. This soft amorphous substance is surrounded and often traversed by the associated oxide of iron.

The plagioclase crystals, even where the twin striation in polarized light is still preserved, contain a greater or less number of small prehnite enclosures; where these are more abundant, the twin striation disappears, and in other parts of the field we find feldspar forms, presenting now a brilliant aggregate polarization, with the characteristic prehnite colors. More commonly the feldspar contours are lost, and we find pseudo-amygdules of prehnite of all sizes, and with the most irregular outlines. These often stand in such intimate connection with prehnite aggregates adjoining feldspar crystals, that it is impossible not to be convinced that the pseudo-amygdules are formed by a growth from a central point outward, attacking the feldspar crystals bodily from the outside. The pseudomorphs proper, on the other hand, result from a change which had countless starting-points in the interior of the crystal. Indeed, we have here a perfect counterpart to the formation of chloritic pseudo-amygdules, and chlorite pseudomorphs after plagioclase, which I have described above, except that here we have a process of prehnitization, while there it is one of chloritization.

A remarkable feature in these sections is the almost total absence of chloritic substance; and it is almost impossible not to see a connection in this with the absence of pyroxene. Indeed, it shows that it is at least quite possible that the extensive change to chlorite in these rocks may be preceded, or mediated, step by step, by the formation of prehnite, which, in its turn, is changed to chlorite, under the co-operation of the decomposition products of pyroxene.

The paragenetic scheme for this rock may be written thus: —



(I.) 1. CHRYSOLITE. 2. PLAGIOCLASE.

(II.) Pseudomorphs of SOFT GREEN MINERAL and SPECULAR IRON after chrysolite.

(III.) PREHNITE in pseudomorphs after plagioclase, and in pseudo-amygdules.

Bed No. 94. — Specific gravity = 2.94. Coarsely crystalline. The most prominent constituent, forming 40–50 per cent of the rock, is feldspar, in narrow, tabular crystals, which are often one inch long. As a rule, the feldspar crystals show, on the cleavage planes, broad bands free from twinning, except where two or three almost microscopic plates of plagioclase are interposed, and a considerable number of the larger crystals are almost wholly free from striation. They are flesh-red from having begun to change. The spaces between these crystals have dark-green and black colors, and, under the microscope with a low power, are found to consist, in part, of a grayish-green, compact mineral, with semi-conchoidal fracture, very soft under the needle-point, and containing veins and fragments of a brilliant black mineral, which, from its streak, and action under the magnet, appears to be magnetite. Other spaces consist of a lighter green mineral, with radiating fibrous structure in segments of spheres, and very soft under the needle. With this light-green mineral are associated aggregated crystalline grains of quartz.

Examining different parts of the specimen, all the constituents are found to envelop long, slender, hexagonal prisms of limpid white apatite.

In uncovered thin sections, under the microscope, we find that the red color of the feldspar is due to a staining on the sides and along the planes of twinning and in fissures. Where the crystals have not become sufficiently altered to be unrecognizable as such, they are not scratched by the needle.

Within the spaces between the feldspar crystals are isolated remnants of augite, in places still transparent, fresh and smoky-brown, but generally surrounded and veined with a slightly translucent grayish-green substance, which is very soft under the needle, and contains veins and grains of magnetite: it is the compact, soft, green mineral mentioned above. The magnetite is here porous in structure, and, when scratched with a magnetized needle, gives a shining black streak, and leaves particles adhering to the needle: facts which indicate that it forms a network of filmy veins intersecting the soft, green substance in all directions, like the cement of a breccia.

The light-green mineral with radiating structure is here seen to be entirely distinct from the grayish-green substance; it is very soft under the needle, and contains no magnetite, but is associated with quartz.

In polarized light some of the feldspar crystals are found to be beautifully marked throughout with thin twin striæ; but these are generally either absent, or only two or three narrow ones are associated with very broad unstriated bands.

Two systems of twinning occur among the feldspar crystals, cutting each other nearly at a right angle. In the greater number of instances, the composition plane is the plane of symmetry; in these double twins, a long, sharply defined crystal will have broad areas free from longitudinal striæ, and apparently of orthoclase, with a few longitudinal narrow bands of a triclinic feldspar intertwinned with it on one side (composition plane = plane of symmetry), while the broad band of supposed orthoclase is crossed here and there by twinned thin laminæ of triclinic feldspar, probably having O for the composition plane. These two sets of triclinic individuals generally meet with a more or less acute angle, but in one instance I obtained a very certain measurement of 89° 30′ — an observation that seems to leave little doubt that we have here a combination of orthoclase with triclinic feldspar, similar, as regards crystallographic structure, to the orthoclase-albite crystals of Harzburg, described by Streng.*

The thin sections of the rock we are describing leave no doubt that orthoclase is an essential primary constituent.

I was able to measure some microscopic crystals directly under the microscope, the angle $O: i\breve{i}$ getting 90° 10′ as the result on an unstriated individual, and 93° 29′ on the striated ones.

Measurements of the entering angle on the cleavage plane, with a reflecting goniometer, gave 93° 24′.

Optical measurements by Des Cloiseaux's method on sections parallel O, and on random sections in the zone $O: i\bar{i}$, determine the feldspar to be oligoclase.

The most prominent appearance of change is the red staining, which is due to the presence of countless minute particles of translucent red or brown iron oxide, which is generally most abundant near the edges, or along the cleavage planes, and in cracks. It is generally more or less present throughout the crystals, though some of these are nearly free from it, except along the above-defined lines.

Where freshest, the feldspar is still clouded, and, under high powers, this is seen to be due to the presence of innumerable minute particles of irregular shape, of a bluish-gray color, and clear, except a narrow dark rim, which seem to be inclosed particles of residuary base.

Besides these, there are large numbers of particles of green substance in warped thick flakes or grains, which seem to aggregate to form wisps or sheaves of long radiating needles. When examined with only the polarizer, these last show an appreciable absorption.

In many crystals, the mass is either clouded through and through with the iron oxide, or is occupied to the extent of from twenty to fifty per cent of its area by the sheaves of radiating green mineral, or both substances are present. The green mineral also forms veins, filling cracks in the feldspar, and is clearly the light-green, soft substance already mentioned. The larger sheaves, occupying the whole thickness of the thin section and the veins, permit some optical determinations on the mineral. Examining it with the polarizer, it shows a marked dichroism, being deep bluish-green, when the longer axis of a needle coincides with the shorter diagonal of the nicol, and light, faintly brown or greenish-brown when at 90°. Between crossed nicols some of the needles become dark when the longer axis is parallel to a nicol-plane; but there often appears to be an angular difference of several degrees. It seems, therefore, considering its softness under the needle, and its optical qualities, to be possibly a monoclinic chlorite. Where the change has gone very far, only fragments of the feldspar remain, surrounded and penetrated by the chlorite and granular quartz.

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* Streng. Feldspath Studien. Leonhard u. Geinitz, Neues Jahrbuch für Mineralogie, 1871, p. 719.

One or two crystals were seen having the contours of feldspar individuals sharply defined, but consisting of granular quartz, showing aggregate polarization, and containing apatite needles and the chlorite. They appear to be pseudomorphs of quartz after feldspar. With the chlorite occurs a yellowish-green chloritic mineral, consisting of closely packed minute spheres, made up of spicules or foliæ, radiating from a centre; they are double-refracting, but show no absorption.

As a rule the augite crystals have disappeared, as such; but, in places, individuals are observed which are in one part perfectly fresh, and in others show the stages of change. Where freshest, they are transparent, almost colorless, with a smoky tinge, and have many irregular fractures, and, in places, parallel cracks, indicating cleavage.

Where they are apparently perfectly fresh, the principal foreign particles they contain are minute ones of a bluish-gray color or, in the larger grains, deep sapphire-blue, and have a narrow, dark rim. These are often in sharply defined rhombic prisms, but more generally of irregular shape, and sometimes connected by hair-like necks. They are almost always arranged in warped planes, in some of which they are much scattered, in others closely packed. A linear arrangement was also observed.

Besides these, the pyroxene contains slender straight rods, which are opaque and black, possibly because of their slender proportions. They are in groups of parallel rods, two groups in places, intersecting each other, and each group being parallel to a different axis. These groups of rods become in places the skeleton of a curious structure; the rods in parallel rows being connected by a bottle-green substance, so as to form parallel uneven planes, like brush-fences, in which the rods represent the posts, and the bottle-green substance the brush, interwoven horizontally, and so closely as to obscure the brush-structure. I can think of no better illustration. In places, this structure exists without any visible black rods.

Where the augite is clouded by a beginning change in the interior, we find these planes of bluish particles and the brush-fence structures to be starting-points, having, in those representing the first stages, only slightly diffused yellowish-green substance; in others, the primary interpositions become more and more obscured, and the planes and arrangement of bottle-green substance are wholly lost in the green mass which penetrates with a serrated front into the fresh augite. This green substance also attacks the crystal from the outside, and along the cracks. Under a high objective the contact resolves itself into a jagged line.

A considerably altered augite crystal is surrounded by a green mass, and more or less broken up by cross-bars of the same substance, while the interior of the remaining augite fragments is often occupied by masses of the green substance which had the interpositions for starting-points.

This alteration product is yellowish-green where very thin, and translucent dirty-green or grayish-green where thicker. It is composed of countless minute stars, indicating radiating structure, and is double-refracting, but shows no absorption for color, and little or none for intensity. The same substance occurs, to a slight extent, in the ruined feldspar with the chlorite, which it sometimes borders.

The green product is associated with magnetite in large and small particles, and, where the change progressed along a crack, the green band has a median line of more or less closely contiguous films and grains of magnetite, which corresponds with the structure, as observed in reflected light on the uncovered sections.

In the rock from the Eagle River, district 94, the augite contains only isolated particles of red ferrite, and then, generally, only in the planes of bluish interpositions, and where these are near the altered portions. But in an almost identical rock, coming from the same zone, north of the Delaware mine, and in which the augite seems to have been wholly changed, an examination in reflected light shows the magnetite surrounded by an opaque mass of red oxide, which, further from the magnetite, resolves itself into a cloud of minute particles of transparent brown, or red ferrite suspended in the green alteration product. Optical measurements on sections of feldspar in the zone $O : i\bar{i}$ in the preparations of this rock gave the same result as in bed 94.

The abundant apatite needles are characterized by their hexagonal sections, or, where cut longitudinally, by their very feeble double-refraction. They often contain many long and straight hair-like black rods, or prisms, ranged strictly parallel to the longer axis of the crystal. Besides these, they contain abundant fluid cavities with dancing bubbles, which did not diminish in size when heated to 95° C.*

They enclose also globular or egg-shaped drops of a greenish-brown substance; on examining these in crystals cut perpendicularly to the longer axis, they were found to be isotrope. There can be no doubt, I think, that these are portions of the original fluid magma, out of which the rock crystallized. We owe to the almost indestructible character of the apatite, and to its lack of double-refraction in one direction, the preservation of this glass, and our ability to recognize its amorphous nature, and thus to obtain the strongest circumstantial evidence of the eruptive origin of these rocks.

The relative ages of the constituents may be represented by the following scheme:—

(I.)	1. APATITE.	2. OLIGOCLASE and ORTHOCLASE.	3. AUGITE and MAGNETITE?
(II.)		c. FERRITE. d. CHLORITE monoclinic? e. QUARTZ.	a. Yellowish-green mineral. b. MAGNETITE?

Bed No. 61.—Among the changes that have resulted in the formation of pseudo-amygdaloids is one which seems to be rare, for I have met with it in only one of the several hundred sections I have cut of these rocks. It is the formation of pseudo-amygdules of analcite on a large scale. The rock is from the *lower part* of bed No. 61 of the Eagle River Section. It is greenish-gray, and breaks with an uneven and somewhat schistose fracture. It is filled with irregularly shaped grains of a white mineral, which fuses, at about three, to a clear glass containing air bubbles; it dissolves in muriatic acid, leaving a deposit of silica, and the filtrate gives with ammonia an abundant precipitate of alumina; in thin sections it is transparent and colorless, and shows its isotrope character by remaining dark during a revolution between crossed nicols. The mineral is undoubtedly analcite. Isolated bands occur, showing very feeble double-refraction.

In thin sections, besides the analcite, the most conspicuous constituent is the green, soft, chloritic substance characteristic of the pseudo-amygdules. In this lie isolated plagioclase crystals, some showing twin striation in polarized light, and others in which this has been lost. Optical measurements on these crystals, on sections cut in the zone $O : i\bar{i}$, indicate that the feldspar is either labradorite or anorthite. A large part of these crystals show an aggregate polarization so closely resembling that of the prehnite pseudomorphs after plagioclase, that there is little doubt that they are partly changed to prehnite. The change to chlorite is evident, in all stages, from chlorite disseminated through the plagioclase, to finished pseudomorphs and pseudo-amygdules.

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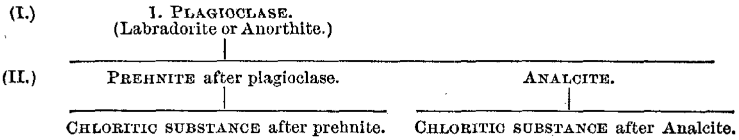
* This is not corrected for the influence of Beck's $\frac{1}{10}$ immersion objective.

The analcite bodies are seen to have no true amygdule-walls, but to penetrate the matrix in the most irregular manner, as in pseudo-amygdules, which they really seem to be. They are traversed by seams of chloritic substance, and are often changed into this to a considerable depth on the sides, besides containing in places groups of the same chloritic substance in the interior. They are also cut by seams of calcite, which also traverse the matrix. The chloritic substance seems in places to have formed finished pseudomorphs after the analcite. The chloritic substance, resulting from change of analcite, is less clear, and much finer textured than that resulting from the change of feldspar.

There are no visible traces of pyroxene. It seems to be quite probable that the analcite is derived from the plagioclase, but I have no data for determining its relative age as compared with that of the prehnite. As a rule, the secondary alkaline silicates form only in the amygdaloids proper, and in veins.


(I.)
I. PLAGIOCLASE.
(Labradorite or Anorthite.)
(II.)
PREHNITE after plagioclase.
ANALCITE.
CHLORITIC SUBSTANCE after prehnite.
CHLORITIC SUBSTANCE after Analcite.

Bed No. 22, of the Eagle River Section, has forty feet of lower zone, and sixteen of pseudo-amygdaloid. Thin sections of the diabase from the lower zone show the plagioclase crystals mostly very fresh. Optical measurements on sections in the zone $O : i\bar{i}$, indicate that the feldspar is labradorite. They are closely surrounded by the younger pyroxene, which is in part very fresh, but in places is changed to its characteristic pseudomorphs. This secondary mineral is better marked here than is usual; it has well-defined, perfect cleavage, strong absorption for intensity, being dark when the cleavage lines coincide with the shorter diagonal of the nicol. Between crossed nicols, darkness occurs when the cleavage lines are parallel to a nicol plane. I observed no instances in which it revolved dark. These characteristics indicate an orthorhombic crystallization. In ordinary transmitted light in thin section, the mineral is olive-green, traversed by iron-stained cracks.

In the pseudo-amygdaloid, while portions of many of the plagioclase crystals are clear and fresh, they are all more or less changed to prehnite, and exhibit every stage from the beginning to the perfect pseudomorph. Prehnite also forms countless pseudo-amygdules, from microscopic size up to one-half inch diameter, which have clearly formed at the cost of the plagioclase. Chlorite occurs in the same manner as the prehnite, forming pseudomorphs after plagioclase, and pseudo-amygdules.

A pseudo-amygdaloid, south of Houghton, contains abundant pseudo-amygdules of prehnite, often an inch or more in diameter. Some of these were found to be partially changed to a coarsely foliaceous chlorite. The prehnite has long-radiating tabular structure. The change has begun between the plates, and in places has wholly altered parts of the prehnite; the pseudomorphs preserve the same radiating structure as the prehnite. In a moderately thin section, the chlorite is dark green and dichroitic, changing from dark green, when its cleavage lines are parallel to the shorter diagonal of the nicol, to a smoky brown when parallel to the longer diagonal.

Small scales pressed between slides in balsam revolved dark between crossed nicols; it is therefore uniaxial.

The feldspar of this rock is apparently either albite or oligoclase, — in all probability the latter; for the highest angle found in ten optical measurements on random sections in the zone $O \mid i\bar{i}$ was 29°.

AMYGDALOIDS.

The uppermost zone — the amygdaloid — in many beds is, in several respects, essentially different from the rest of the rock. In these instances, the matrix has a much finer texture, often quite aphanitic, even where the lower and pseudo-amygdaloid zones of the same bed are quite coarse grained and distinctly crystalline. The amygdules have generally spherical or ovoidal forms, filling cavities with sharply defined walls. In some rare instances, the amygdules are long and cylindrical, and arranged perpendicular to the plane of bedding.

In thin sections, the difference between the texture of the matrix and the texture of the lower zone of the same bed is very apparent. While the primary constituents, when preserved, do not differ apparently in quality, they are of much smaller size, and sometimes show an arrested development and microfluidal structure. Here, too, the amygdules have sharply defined, evenly curved contours, differing wholly in this respect from the pseudo-amygdules of the lower zones of the same bed. Often also the primary minerals of the matrix are much more minute in the immediate neighborhood of the amygdules than away from them, and their disposition seems also to have been influenced by the presence of the amygdaloidal cavity. The color of the amygdaloid matrix varies between different shades of dark greenish brown, chocolate-brown, dull light green, and brilliant light green. All these colors often occur in the same bed, or even in the same specimen, and, like the differences in hardness, — from 3 to 7, — are due to metasomatic changes.

There is one variety — the *scoriaceous amygdaloid* — which occurs less often than the others, but is always strongly characterized. It consists of true amygdaloid and sandstone curiously associated. The amygdaloid is in contorted forms, from an inch or less to several feet in size, with a sharply defined, often wrinkled, surface, and wholly enveloped with sandstone. The glaciated surface of the outcrops of this rock often present at the first glance the appearance of a conglomerate, in which here the amygdaloid, there the sandstone, appears to form the cement. Thin sections show that we have to do with a true sandstone of quartz and feldspar derived from quartz porphyry, and identical with that which forms the great sandstone beds of the copper series; and that the other constituent is a true amygdaloid.

The conditions, as studied on the ground, indicate that *these* beds are volcanic scoriæ, buried in the littoral sand. To this variety belong the famous "Ash-bed" of the Copper-falls and Phœnix mines, and the beds worked in the Hancock and South Pewabic mines.

Bed No. 87 of the Eagle River Section has an actual thickness of 154 feet, of which 7 feet belong to the amygdaloid, 23 to the pseudo-amygdaloid, and 124 to the lower zone. The lower zone has a rather fine grain, and shows to the naked eye only green crystals of feldspar, and irregular-shaped small spots of dark green chlorite. These constituents give to the rock a dirty gray green color, spotted with dark green. The powder yields some magnetite.

In thin sections, from the lower zone, we can distinguish plagioclase, augite, an impellucid, dirty gray, unindividualized substance, containing often radiating colorless needles of a feebly double-refracting substance, perhaps apatite; besides these, there are the abundant secondary products.

The augite, which is the least altered constituent, is generally highly fissured, and in places altered to its characteristic chloritic product, more or less associated with red and black iron stains, which are possibly the source of some of the magnetite.

The plagioclase retains in many places its twin-striation in polarized light, but, even where freshest, it contains many tufts of radiating fibres or laminæ of chlorite, and the sections show that a large part of the feldspar has been changed to pseudomorphs and pseudo-amygdules of chlorite. Optical measurements on random sections in the zone $O : i\bar{\imath}$ give only low angles that may belong to either albite or oligoclase, here undoubtedly the latter. The specific gravity of the rock is 2.73, which also points to oligoclase.

The external appearance of the middle zone differs from that of the lower in having a rather coarse grain, and in that the feldspar crystals are pink and of all sizes, from $\frac{1}{5}$ inch down, while the very irregularly shaped masses of dark green chlorite often reach $\frac{1}{3}$ inch in diameter.

In thin sections from the middle zone, I found the pyroxene wholly represented by its characteristic pseudomorphs with iron-stained cracks; the plagioclase is also much more altered.

The formation of the chlorite pseudo-amygdules after the manner already described is beautifully illustrated in this rock.

This chloritic substance appears, under a glass, both as a compact and as a very finely scaley, dark-green mineral. The hardness of the compact portions is 2.5. It fuses at 3–3.5 to a dull black magnetic globule. It dissolves in muriatic acid, leaving pulverulent silica, and the solution contains alumina, protoxide of iron, and magnesia; an appreciable amount of both potash and soda was found, both after boiling in water and in the acid solution. (See complete analysis of this mineral.)

In the thin sections, the radiating laminæ are decidedly dichroitic, being straw-yellow when the longer direction coincides with the longer diagonal of the nicol, and bluish green when parallel to the shorter diagonal. Between crossed nicols it seems to be uniaxial, for the crystals become dark when the longer direction coincides with a nicol plane, and portions were found which revolved dark. Its appearance in polarized light corresponds very closely with that of a section of diabantite* which Mr. Hawes kindly sent me; but it differs in its hardness, diabantite being only 1.

The following analyses (made for me by Mr. Woodward, in the Laboratory of the Sheffield Scientific School) are of specimens from the lower and middle parts of bed No. 87, and of the chloritic mineral forming the pseudo-amygdules.

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* On Diabantite. Geo. W. Hawes, Amer. Jour. Sci., June, 1875, p. 454.

Bottom of Bed 87.		Middle of Bed 87.	
SiO_2	46.32	SiO_2	49.20
Al_2O_3	15.95	Al_2O_3	16.00
Fe_2O_3	2.86	Fe_2O_3	3.03
FeO	8.92	FeO	7.10
MnO	.89	MnO	1.17
CaO	10.28	CaO	3.44
MgO	4.08	MgO	6.98
TiO_2	2.78	TiO_2	2.26
K_2O	1.23	K_2O	1.31
Na_2O	3.56	Na_2O	5.05
H_2O	3.25	H_2O	4.51
	100.12		100.05

Both rocks are too highly altered to permit a calculation of the mineral constituents.

Chloritic pseudo-amygdules in bed No. 87.				Analysis of chloritic amygdules from the rock under the Quincy copper-bearing bed. By MacFarlane. (Geol. of Canada, 1866, p. 154.)	
		Per cent. ÷ at. Weight.			
SiO_2	29.56	.492	2.1	SiO_2	30.59
Al_2O_3	22.41	.207	1.	Al_2O_3	26.07
Fe_2O_3	4.38	.027			
FeO	22.20	.308	2.5	FeO	22.01
MnO	1.18	.016			
CaO	2.16	.038		CaO	1.92
MgO	8.36	.209		MgO	12.36
K_2O	.15	.001			
Na_2O	.49	.008			
H_2O	9.07	.503	2.1	H_2O	7.23
	99.89				100.18

The mineral of the pseudo-amygdules approaches more nearly to delessite than to any other of the chlorites. I have added an analysis of a similar pseudo-amygdaloidal chlorite from another part of the district.

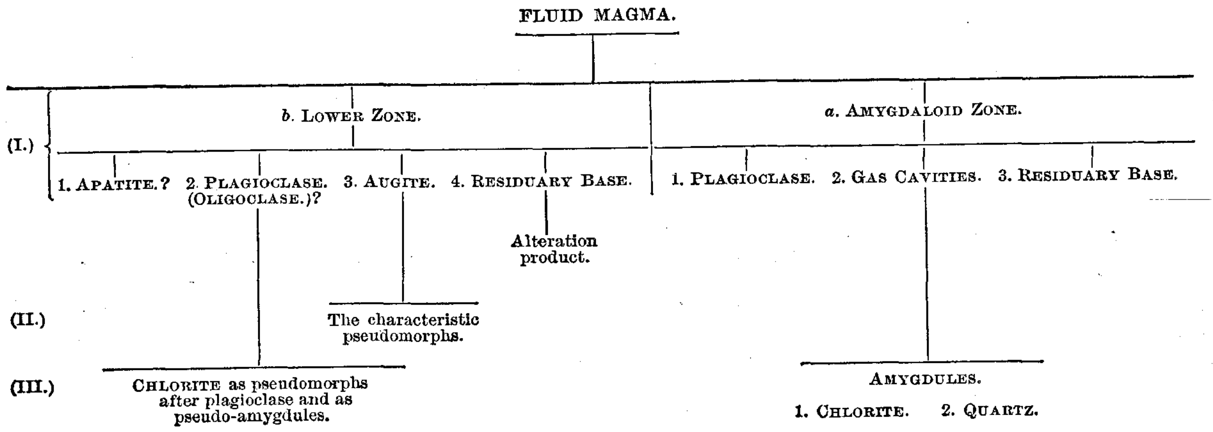
The amygdaloid of this bed is compact — almost aphanitic — in texture, and is reddish-brown to greenish brown in color. Nearly one-fifth of the rock is occupied by amygdules averaging $\frac{1}{8}$ inch diameter, but sometimes much larger, and consisting chiefly of quartz covered by, and more or less impregnated with, chlorite. In the thin sections, the matrix consists chiefly of small plagioclase crystals, in which the twinning is still more or less apparent in polarized light, but they show marked alteration. The spaces between the plagioclase crystals are filled with an impellucid mass of brown particles of iron oxide, and confused, often radiating, long, slender, colorless, translucent crystals. These crystals are feebly polarizing, and at first sight appear like apatite, but I observed no hexagonal sections. They often start out from the end of a feldspar crystal and radiate from this, which, taken in connection with their appearance, renders it quite likely that they are feldspar microlites arrested during development into crystals. I noticed no pyroxene, and only very isolated apparent pseudomorphs after it. The impellucid substance between the feldspar microlites contains much soft, green, chloritic substance. As pyroxene in these rocks shows itself to have always crystallized after the feldspar, we should, perhaps, not expect to meet with it where the rock solidified before the feldspar microlites had united to form finished crystals.

The amygdules have almost always sharply defined, smooth walls, and are then bordered by a circumference of the unindividualized substance with its feldspar microlites, and these latter are then arranged in a manner with reference to the amygdule that seems to clearly indicate the exertion of a force by the cavity on a surrounding semifluid medium.

In places, the sharply defined outline and the unindividualized border are missing, and the formation of a pseudo-amygdule has taken place, often more or less enveloping the true amygdule.

The amygdules consist chiefly of quartz in crystalline aggregates, filling the interior, and surrounded by a mural lining of chlorite, consisting of long, thin narrow plates, which are either orthorhombic or uniaxial, and which bristle toward the interior. Long radiating tufts of these plates penetrate far into the interior of the pellucid quartz crystals, indicating that the chlorite lining is older than the quartz.

Paragenesis: —


FLUID MAGMA.
b. LOWER ZONE.
a. AMYGDALOID ZONE.
(I.)
1. APATITE.?
2. PLAGIOCLASE. (OLIGOCLASE.)?
3. AUGITE.
4. RESIDUARY BASE.
1. PLAGIOCLASE.
2. GAS CAVITIES.
3. RESIDUARY BASE.
Alteration product.
(II.)
The characteristic pseudomorphs.
(III.)
CHLORITE as pseudomorphs after plagioclase and as pseudo-amygdules.
AMYGDULES.
1. CHLORITE.
2. QUARTZ.

Bed No. 69 of the Eagle River Section consists of fifty-six feet of the lower zone, eleven feet of pseudo-amygdaloid, and six feet of amygdaloid.

The lower zone is a fine-grained, dirty-green rock with uneven fracture. It is easily scratched; has specific gravity 2.87–2.95, and the powder yields a little magnetite.

The thin sections resemble those of the lower zone of bed 87. The plagioclase is much altered, — containing in the freshest many tufts of chlorite — and is often represented only by pseudomorphs of chlorite, and in places these are merged into chlorite pseudo-amygdules.

The augite is in part very fresh, in part changed to its characteristic pseudomorph.

The amygdaloid is a very compact, hard rock, with subconchoidal fracture. It consists of very irregularly mixed brown and green portions, both hard, the brown abounding in amygdules, from one-third inch diameter down, chiefly of prehnite; often of prehnite as an outer member, and a central filling of quartz in some, in others calcite. The green contains fewer apparent amygdules.

Thin sections of the brown part show the sharp outlines of comparatively large porphyritic feldspar crystals, and of countless long slender feldspar microlites separated by an opaque brown substance. These feldspar forms are now occupied by brilliantly polarizing aggregates of prehnite.

Splinters of this brown matrix fuse in the flame of an alcohol lamp.

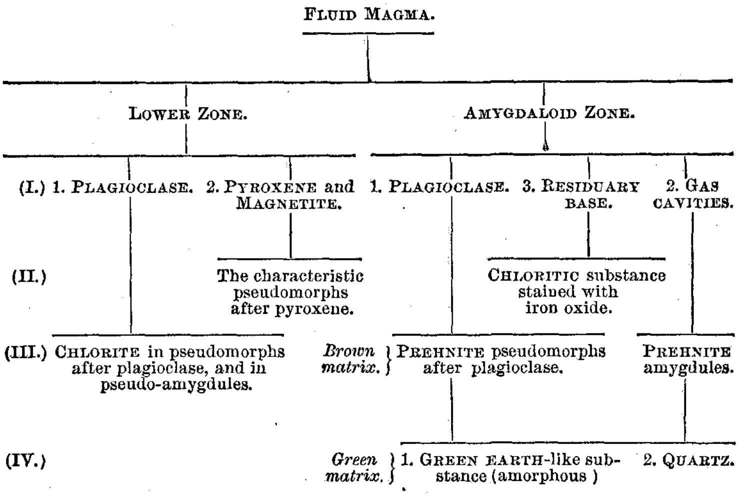
Some of the feldspar forms contain a large amount of a soft, light-green, seemingly amorphous mineral, which is, probably, pseudomorphous after prehnite; the rest of the pseudomorph in these cases *seems* to be quartz.

The amygdules have very sharply defined contours, and form brilliantly polarizing aggregates of prehnite. Quartz occurs in seams which cut through the prehnite of the matrix, and of the amygdules.

An examination of thin sections of the green parts shows that they are derived from the brown. They consist still to a great extent of prehnite, and many pseudomorphs of this after the feldspar are visible; but it is everywhere more or less changed to the light-green, soft substance (of which some was seen in the brown variety), and considerable areas of the field are wholly changed to this substance, which is thoroughly cut up by curving cracks of irregular shape and size, which are evidently due to contraction, and are now filled with quartz. But little of the brown staining seen in the brown variety is present here: the iron oxide causing it has, perhaps, gone towards forming the green-earth-like alteration product of the prehnite. Splinters of this variety show under the loupe by transmitted light, nearly opaque, light-green portions, separated by transparent white. The white fuses in the flame of an alcohol lamp.

The specific gravity of the brown part, taken where there were only very small amygdules, was 2.80; and that of the green, 2.83.

Paragenesis: —


Fluid Magma.
Lower Zone.
Amygdaloid Zone.
(I.) 1. Plagioclase.
2. Pyroxene and Magnetite.
1. Plagioclase.
3. Residuary base.
2. Gas cavities.
(II.)
The characteristic pseudomorphs after pyroxene.
Chloritic substance stained with iron oxide.
(III.) Chlorite in pseudomorphs after plagioclase, and in pseudo-amygdules.
Brown matrix.
Prehnite pseudomorphs after plagioclase.
Prehnite amygdules.
(IV.)
Green matrix.
1. Green earth-like substance (amorphous)
2. Quartz.

HANCOCK MINE.

A specimen of chocolate-brown amygdaloid, from the halvans of the Hancock mine, contains beautiful amygdules of a dark green finely scaly chlorite, each one surrounded by a narrow ring of quartz. The matrix contains some grains that seem to be pyroxene, and some of the feldspar crystals, though much altered, still show twin-striation. But whole groups of the feldspar have been replaced by quartz in such a manner, that the quartz polarizes the light as an integral individual throughout the area of each group. The pyroxene grains and pseudomorphs within the areas of these groups have not been changed to quartz.

The chlorite of the amygdules is highly dichroitic, being green when the longer axis of the laminæ coincides with the shorter diagonal of the nicol, and yellow when perpendicular to this. Portions revolve dark between crossed nicols, as did also scales pressed in balsam between glass. It is therefore uniaxial. The quartz which forms the outer layer of the amygdules is connected with, and really forms part of, veins which traverse both the matrix and the chlorite of the amygdules. In doing this, it penetrates between some of the laminæ, and encloses others in a manner that proves it to be younger than the chlorite.

Bed No. 64. — The amygdaloid of bed 64 of the Eagle River Section has about sixty per cent of its volume occupied by amygdules, sometimes wholly prehnite, sometimes an outer layer of white prehnite, and a central filling of calcite. The matrix is chocolate-brown, and has a crystalline texture wholly foreign to the melaphyres, and more resembling that of a fine-grained, somewhat oxidized spathic iron ore. Its hardness is 6; fusibility 2–2.5; it dissolves in muriatic acid, leaving pulverulent silica, and the solution contains abundance of alumina and lime; in thin sections it is seen to be clearly orthorhombic, and polarizes the light with the same colors as prehnite, which it undoubtedly is. In thin sections, by ordinary light, the first things we see are the characteristic outlines of plagioclase crystals, filled with a limpid colorless substance, while all the interstitial spaces are filled with a less clear substance, colored brown by countless particles of iron oxide.

Examining it between crossed nicols, a remarkable change takes place. The plagioclase outlines are still sharply defined by the abundance of particles of iron-oxide suspended in the interstitial substance; but every thing except these brown particles is changed to prehnite. In places, the feldspars are each occupied by a fine-grained aggregate of prehnite; but often the latter mineral has crystallized more freely a group of long, radiating, tabular individuals, reaching with brilliant red and green colors across whole groups of plagioclase crystals and the interstitial spaces, and sometimes well into an amygdule without a break in the integral polarization of each plate.

Below the amygdaloid of bed 64, just described, there are several beds (with an aggregate thickness of twenty feet) forming apparently a transition into a pseudo-amygdaloid. The rock of these beds has a more or less light green color, a compact, aphanitic matrix, in which lie abundant amygdules $\frac{1}{8}$ inch and less in size. Many of these are filled with prehnite; as many more are cavities lined with rosy crystals of adularia, while others contain both of these minerals. The feldspar crystals in the cavities are sharply defined prisms, terminated at the free end with the basal plane. In thin sections, the matrix contains much pyroxene unaltered, except that it is much broken.

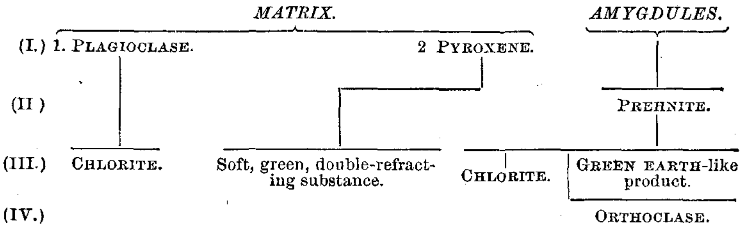
The plagioclase is all much altered, and a considerable proportion of the crystals is changed to chlorite. Where they are still colorless, they have lost the twin striation in polarized light.

The amygdules have generally, not always, sharp outlines. Some are of unaltered prehnite. Others, which have evidently consisted of prehnite with long radiating structure, are more or less altered to a soft, homogeneous, impellucid, green substance, which seems to be structureless, or to polarize the light only very feebly. It is a frequent alteration product of prehnite in these rocks. Some of the amygdules are wholly changed to chlorite. In most of the amygdules containing altered prehnite, the orthoclase occurs, showing aggregate polarization, and intimately associated with small fragments of prehnite, partially altered to the green substance; and while these are scattered through the orthoclase aggregate, they show in polarized light that they are merely remnants of a formerly continuous radiating mass of prehnite, the rest of which has been changed to orthoclase. As a rule this change has been accompanied by a large diminution of volume, resulting in a central empty cavity, into which the feldspar crystals project freely crystallized, and show there integral polarization.

The appearances seem to indicate that the pseudomorphs of orthoclase were formed after the partial destruction of the prehnite.

The pyroxene of the matrix has, in places, been altered to a bright yellowish-green, soft, double-refracting substance: none of the characteristic pseudomorphs were seen.

The paragenesis, in so far as it is determinable, is —



The melaphyre proper, which forms the lower member of bed No. 64, is a dark green, almost black, cryptocrystalline rock, which is easily scratched with the knife. Under the microscope, it is found to consist chiefly of plagioclase in very small crystals, a soft, green mineral, probably pseudomorphous after olivine, minute grains of augite, and occasional small, often wedge-shaped, occurrences of a green soft substance, occupying the interstices between feldspar crystals.

The feldspar appears, from optical measurements in the zone $O : i\bar{\imath}$, to be anorthite.

The augite is apparently fresh, and is very subordinate in quantity and size of individuals. Next, as regards quantity, to the feldspar, is the soft, green mineral. It is in rather rounded grains, suggesting rectangles with the corners rounded off, and elongated hexagons. The contours are not broken by the feldspar or augite crystals, from which they would seem to be the oldest constituent. They are strongly marked by thick, parallel, transparent, colorless lines, indicating open cleavage-cracks, show strong absorption for intensity, and become dark between crossed nicols when these lines are parallel to one of the nicol undulation-planes. Many were seen which showed no parallel lines, and these were, probably, cut parallel to the plane of cleavage, but none of them revolved dark between crossed nicols; the mineral is, therefore, probably orthorhombic. There can be little doubt that this mineral is pseudomorphous after olivine. Its contours are identical with those of the olivine in 108. The parallel structure is there represented by the tendency to a fibrous structure; and both the alteration product and the fresh olivine of the same individual are dark, when the direction of these fibres is parallel to one of the nicol undulation-planes.

There are also present many small pseudo-amygdules, filling the wedge-shaped interstices between the feldspars.

The annexed analyses of specimens from the three members of bed No. 64 serve to throw some light on the primary mineral constitution of the rock, and on its alteration. The calculation of the primary minerals in the fresher, bottom rock, can be only roughly proximative, while the optical measurements indicate a predominance of anorthite. The analysis confirms this, and points to the presence of a little soda, or soda-lime feldspar.

In the analysis from the middle, or amygdaloidal region, we find a loss of one-third of the soda; and a gain in lime, which marks the alteration of the feldspar to prehnite. The potash belongs to the orthoclase, which is pseudomorphous after prehnite.

The analyses were made for me by Mr. Woodward, of the Sheffield Scientific School.

Melaphyre. Lower Zone of Bed 64.	Per cent, ÷ At. Weight.	Orthoclase.	Albite.	Anorthite.	Pyroxene.	Magnetite.	Residue representing the product after olivine and the pseudo-amygdules.	Middle Zone (pseudo-amygdaloid), Bed 64.	Per cent, ÷ At. Weight.	Prehnitized Amygdaloid of Bed 64.	Per cent, ÷ At. Weight.
SiO_2 . . 47.74	.795	.006	.186	.260	.170		.173	SiO_2 . . 42.83	.714	SiO_2 . . 42.71	.711
Al_2O_3 . . 16.75	.162	.001	.031	.130				Al_2O_3 . . 16.58	.161	Al_2O_3 . . 14.93	.145
Fe_2O_3 . . 2.55	.015					.015		Fe_2O_3 . . 4.42	.028	Fe_2O_3 . . 7.45	.046
FeO . . 6.31	.087				.024	.014	.049	FeO . . 3.81	.052	FeO . . 3.48	.048
TiO_2 . . 1.02	.012					.012		TiO_2 . . 1.36	.016	TiO_2 . . 1.29	.015
MnO . . .52	.007							MnO . . .87	.012	MnO . . .22	.003
CaO . . 11.40	.203			.130	.073			CaO . . 14.11	.251	CaO . . 22.76	.406
MgO . . 8.32	.208				.073		.135	MgO . . 6.96	.174	MgO . . 2.70	.067
Na_2O . . 1.93	.031		.031					Na_2O . . 1.29	.020	Na_2O . . .54	.008
K_2O . . .14	.001	.001						K_2O . . 1.39	.014	K_2O . . .04	.0004
H_2O . . 2.73	.151						.151 ?	H_2O . . 6.48	.360	H_2O . . 3.56	.197
Specific gravity 2.69. Augite taken as $\dot{M}g : \dot{C}a : \dot{F}e = 3 : 3 : 1$.								Specific gravity 2.85.			

On the Mesnard property, fifty feet west of the Houghton Conglomerate, there is a pseudo-amygdaloid melaphyre, which I mention here, because the changes that have taken place in it belong more properly in the stage of development found most commonly among the amygdaloids.

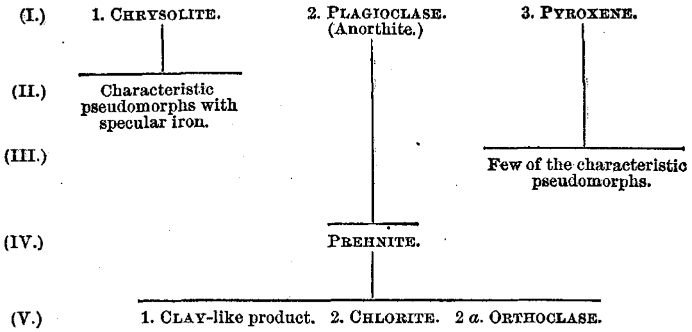
In a fine-grained, greenish-brown matrix lie pseudo-amygdules, which sometimes attain a diameter of one inch or more. On breaking these, we find a soft, dark-green compact substance, speckled with white and red, the latter from orthoclase, which, in places, shows the combination of a prism with the basal plane.

In thin sections, the matrix is found to be of the same type as the "greenstone," though coarser grained, and less unaltered. Here, as there, the chrysolite grains are crowded into the spaces intervening between relatively large areas, each of which consists of a pyroxene individual, enclosing many plagioclase crystals. The constituents are plagioclase crystals, generally very fresh, much unaltered pyroxene, and relatively few of the characteristic pseudomorphs after this, but many after chrysolite, associated with considerable specular iron, also from the chrysolite in part. Besides these, there are numerous pseudo-amygdules of chlorite and some pseudomorphs of chlorite after plagioclase. Optical measurements in the zone $O: i\bar{\imath}$, appear to indicate anorthite.

The large pseudo-amygdules were undoubtedly once prehnite; but of this there now remain three products. Rounded areas of a soft, opalescent, white, translucent substance show a minutely scaly aggregate polarization, and revolve equally bright, without apparent change, between crossed and parallel nicols: this is probably a clay. These areas are fringed with a hard, white, or pink-white mineral, showing granular aggregate polarization, and belonging to the orthoclase. Often this feldspar occupies a large part of the white areas, and is then distinguished from the soft substance by its coarse aggregate polarization of light. These areas of soft, white substance and orthoclase are surrounded by a green chloritic mineral, which is made up of minute spheres with radiating structure, and which resembles that of the smaller pseudo-amygdules in the matrix.

The appearance of the sections is such, that the soft, white alteration product of the prehnite seems to be the starting-point for the formation of both the orthoclase and the chlorite.

Paragenesis: —


(I.)
1. CHRYSOLITE.
2. PLAGIOCLASE. (Anorthite.)
3. PYROXENE.
(II.)
Characteristic pseudomorphs with specular iron.
(III.)
Few of the characteristic pseudomorphs.
(IV.)
PREHNITE.
(V.)
1. CLAY-like product.
2. CHLORITE.
2 a. ORTHOCLASE.

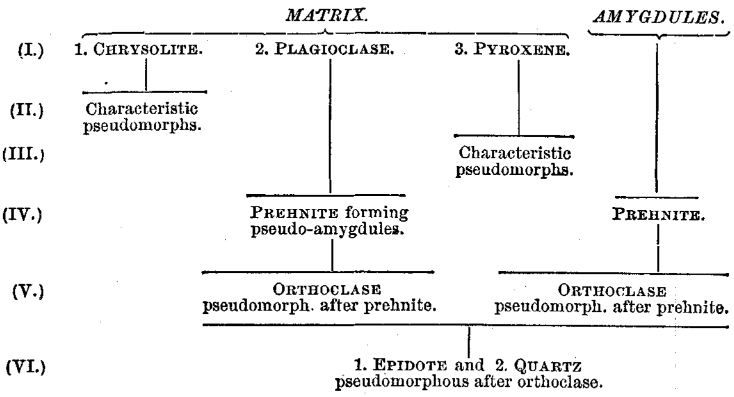
One hundred and fifty-eight feet west of the Isle Royale copper-bearing bed, on the Sheldon and Columbian property, there is a very interesting amygdaloid. It has an aphanitic, brownish-green matrix, abounding in amygdules of rather irregular shape, but with sharply defined contours. These vary from microscopic size to several inches, the larger ones having the most irregular forms. The larger ones consist of intimately associated white prehnite, granular red orthoclase, quartz and epidote. The smaller ones, containing little prehnite visible with the naked eye, have an outer member of red orthoclase freely crystallized in the middle, and have in the centre, quartz, or epidote in small crystals, or both.

The feldspar crystals are profoundly altered, and present now mere honeycombed forms.

In thin sections, the orthoclase pseudomorphs after prehnite still polarize the light; but such of the feldspar substance as remains is rendered impellucid by many suspended particles of iron oxide; and the honeycomb cells are filled with quartz, which also occupies the central area of the amygdule. Epidote, in short and long prisms, occurs within the feldspar forms, and often starting in the cavernous interior projects beyond the end of a feldspar, and is there wholly enclosed in quartz; it also occurs apparently wholly suspended within quartz individuals.

In the matrix, relatively few of the plagioclase crystals show a just apparent twin-striation in polarized light. In most of them it has disappeared. There is no unaltered pyroxene, but there are some of the characteristic pseudomorphs after that mineral, and others after chrysolite. The interstitial spaces are filled with an iron-stained, soft, slightly green substance, somewhat resembling that of the pseudo-amygdules. Besides this, the matrix is in many places filled with pseudo-amygdules, — many wholly occupied by epidote aggregates, others by orthoclase with epidote. These appear to have formed at the cost of the plagioclase. Some of the plagioclase crystals, especially near these pseudo-amygdules, are partially changed to aggregates of epidote.

Paragenesis: —


MATRIX.
AMYGDULES.
(I.)
1. CHRYSOLITE.
2. PLAGIOCLASE.
3. PYROXENE.
(II.)
Characteristic pseudomorphs.
(III.)
Characteristic pseudomorphs.
(IV.)
PREHNITE forming pseudo-amygdules.
PREHNITE.
(V.)
ORTHOCLASE pseudomorph. after prehnite.
ORTHOCLASE pseudomorph. after prehnite.
(VI.)
1. EPIDOTE and 2. QUARTZ pseudomorphous after orthoclase.

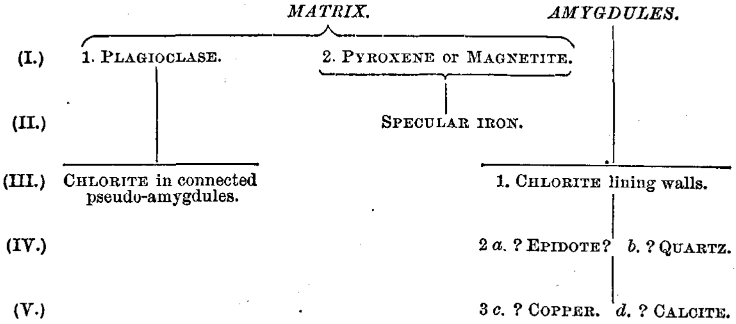
Spike Amygdules. — I have before me a specimen from one of the mines working in an amygdaloid. It has an even purple-brown matrix, in which occur isolated porphyritic crystals of red feldspar sometimes $\frac{1}{10}$ inch in size. Some of these may be orthoclase, as they show no twin striation on the cleavage planes in reflected light, while others are evidently triclinic. The loupe discovers numerous minute flakes of specular iron. Besides these, there are many very small round amygdules of chlorite, each forming a sphere with radiating structure, while others, always larger ones, have an outer ring of this chlorite, and a central filling of calcite, in which the uninterrupted cleavage indicates a single individual. The rock contained, also, many irregularly cylindrical cavities, often 5 inches long, and $\frac{1}{10}$ to $\frac{1}{3}$ inch thick, running parallel to each other and perpendicular to the plane of bedding. These are now filled chiefly with metallic copper, in continuous, more or less solid masses, which break out from the rock in the form of rough-sided spikes. Many of the small, round cavities adjoining a long one are connected with it and filled with copper, so that, when the large spike is detached, its sides have numerous small copper amygdules joined to it by a neck.

Besides the copper, these cavities contain quartz, calcite, and chlorite. The chlorite forms the oldest member, lining the walls of the former cavity; quartz, more or less massive, with frequent perfect prisms, comes next. Younger than the quartz are copper and calcite. The large calcite individuals break out with perfect casts of the quartz prisms, and the copper spikes are indented on the sides with impressions of the quartz crystals, showing every detail, including the horizontal striation of the prisms, with all the sharpness of an electrotype. The relative ages of the copper and calcite does not appear. Both the copper and calcite form fine seams, traversing the matrix; but they were not observed together in the same crack; and copper flakes abound in the cleavage of the macroscopic feldspar crystals.

On the sides of the spike-cavities, after removing the copper, were found groups of very small, hard, green crystals, which seem to be epidote. They are younger than the chlorite, and older than the copper; for this bears sharp impressions of the crystal groups.

In thin sections of the matrix we find the isolated macroscopic crystals of plagioclase, and more rarely others, which show no twinning, and are perhaps orthoclase; and these are undoubtedly primary constituents. Aside from these, the matrix consists of small plagioclase crystals, more or less altered, but generally showing the twinning in polarized light, and lying in an almost continuously connected mass of pseudo-amygdaloid chlorite, which is frequently obscured by aggregated flakes of specular iron and brown stains.

Paragenesis: —

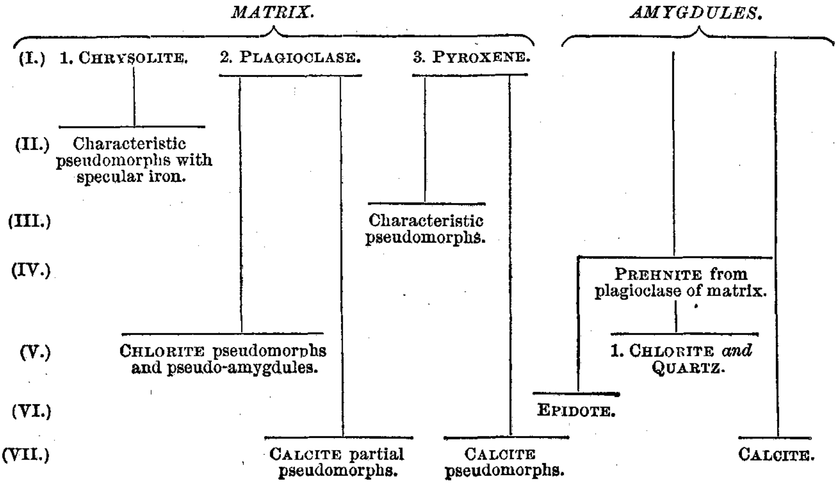


A purple-brown amygdaloid from the Sheldon and Columbian property has a very fine-grained matrix. It contains, 1st, spherical amygdules, $\frac{1}{3}$ to $\frac{1}{2}$ inch in size, of prehnite, more or less altered to a pliable, white, chalky substance; 2d, spherical amygdules, $\frac{1}{3}$ inch and less, of dark, green chlorite, with finely scaly texture, enclosing a little quartz; 3d, spherical amygdules, with a central filling of the white altered prehnite, of very irregular shape, and surrounded by epidote, with comparatively coarse crystallization, and seemingly pseudomorphous after prehnite; 4th, an amygdule of epidote, containing two irregular central masses of calcite separated by a partition of epidote: the calcite adapts itself to the surfaces of the freely crystallized faces of the epidote crystals; 5th, amygdules filled with epidote alone in crystalline masses radiating from several circumferential points into the interior.

In thin sections of the matrix, the plagioclase crystals retain, in many instances, tolerably defined twin striation in polarized light; but nearly all of them are partially, others wholly, changed to chlorite, which forms connected pseudo-amygdules throughout the matrix. Numerous characteristic pseudomorphs, with specular iron, after chrysolite occur. But with the exception of some characteristic pseudomorphs, all the pyroxene is now represented by calcite pseudomorphs, each showing *integral* polarization. Some of the plagioclase crystals seem, also, partly altered to calcite with aggregate polarization. The change from pyroxene to calcite was apparently here, as in the melaphyre out of Mabb's vein, the last transformation.

If we consider the prehnite to be the oldest member of the amygdules, as we have heretofore found it to be, and assign its age to the period of formation of the pseudo-amygdules out of the plagioclase, we shall be justified in considering the chlorite and quartz amygdules as produced before the formation of the calcite pseudomorphs after pyroxene. For we have seen the change of prehnite amygdules to chlorite and quartz, where the pyroxene of the matrix was almost intact. The pseudomorphs of epidote after prehnite were probably mediated by the infiltration of the dissolved products of alteration of the pyroxene.

If this interpretation be right, the paragenetic tree should be:—



MATRIX.
AMYGDULES.
(I.)
1. CHRYSOLITE.
2. PLAGIOCLASE.
3. PYROXENE.
(II.)
Characteristic pseudomorphs with specular iron.
(III.)
Characteristic pseudomorphs.
(IV.)
PREHNITE from plagioclase of matrix.
(V.)
CHLORITE pseudomorphs and pseudo-amygdules.
1. CHLORITE *and* QUARTZ.
(VI.)
EPIDOTE.
(VII.)
CALCITE partial pseudomorphs.
CALCITE pseudomorphs.
CALCITE.

Ossipee Amygdaloid. — A specimen from an amygdaloid, 725 feet east of the Calumet Conglomerate, on the Ossipee location, has a compact, amorphous-looking, soft, dark-green, and brown matrix. It contains numerous very irregular spots and patches of prehnite, from the size of a pin-head to several inches. These are in places more or less altered — some wholly — to a light and dark-green, soft, chloritic substance, which has conchoidal fracture, and appears amorphous, even under a strong loupe. In some of the amygdules this chloritic product is associated with intermingled calcite; in others, with quartz. The larger patches of prehnite contain druses, some of which were formed by removal of prehnite substance, while others seem to represent former cavities, and are now lined ¼ inch thick with reniform prehnite, now much altered and easily scratched. These druses are half filled with small well-formed crystals of epidote and orthoclase, and isolated ones of copper. These sit upon the altered prehnite. The epidote, in minutely granular aggregates, penetrates the reniform masses of prehnite in such a manner as shows it to be pseudomorphous after the latter. The orthoclase crystals, in places, sit directly on the altered prehnite; in others, on the epidote crystals. The copper is also younger than the epidote. A thin section shows that the matrix has been prehnitized, while some of this prehnite is still present: a large part of it is changed to the pseudo-amygdaloidal chlorite, still preserving the plagioclase outlines.

A small specimen from an unknown locality in Ontonagon County, has a very soft purple matrix, which is impregnated with aggregates and crystals of epidote. It contains also irregular druses, lined with small epidote crystals, on which sit crystals of copper; and on these again, as well as on the epidote, crystals of orthoclase.

Bunches of minute, light, gray-green prisms of a very soft mineral sit on the epidote crystals. In so far as external appearance and all the optical tests that can be applied under the microscope go, this mineral is identical with that which, alone and included in quartz, forms amygdules in the cupriferous amygdaloid of the Pewabic and other mines. That from the Pewabic mine was analyzed by Macfarlane,* with the following result: —

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* Geol. of Canada, 1866, p. 153.

Silica	46.48
Alumina	17.71
Protoxide of Iron	21.17
Lime	9.89
Magnesia	trace
Alkalies by difference	1.97
Water	2.78
	100.00

Mr. Macfarlane says: "It fuses before the blowpipe to a black, slightly magnetic glass. On ignition it changes to a light yellow color, losing 0.4 per cent of its weight. It is decomposed by hydrochloric acid, and the resulting solution contains protoxide, as well as peroxide of iron." In the analysis the iron was calculated as protoxide, and the difference between it and peroxide put down as water.

It seems possible that, at least, much of the lime was due to intermingled calcite; for I found, under the microscope, calcite particles in the powder scraped carefully from the amygdules of the Pewabic occurrence.

Between crossed nicols this mineral appears to be orthorhombic; with one nicol it shows feeble dichroism. It is, probably, one of the many minerals for which we have as yet no better general name than green-earth.

There can be little doubt that here, as in the specimen from the Ossipee amygdaloid, the matrix has been prehnitized, and the prehnite changed to pseudo-amygdaloidal chlorite, while the prehnite of the amygdules, small and large, was changed to epidote and orthoclase.

Amygdaloid from Section 8, *Town.* 47, *Range* 45. — Some interesting specimens, illustrating the change of prehnite into epidote and orthoclase, were collected by me in the Southern Copper Range at 400 paces north of the east ¼ post of Section 8, Town. 47, Range North, 45 west, in the southern part of Ontonagon County. The rock is an amygdaloid with a fine grained gray-green matrix. It contains numerous small amygdules of compact dark-green pseudo-amygdaloidal chlorite, and many large spike amygdules sometimes two or three inches long, and ¼ inch or less in diameter. Some of these are filled wholly with epidote, others with epidote as the older, and quartz as the younger member. In some, one portion of the spike consists of epidote, while the rest is the same dark-green chlorite that forms the small amygdules. More rarely the larger amygdules consist of red orthoclase, generally with quartz as a younger member. None of the amygdules are drusy. The epidote is in massive acicular aggregates, its structure converging towards the centre. The orthoclase and quartz are in granular aggregates.

In a thin section of the matrix we see that the greater part of the plagioclase still shows the twin striation in polarized light. There is some unaltered pyroxene, but none of the characteristic pseudomorphs after this. The section contains some pseudo-amygdules of the characteristic chloritic mineral; and others formed of granular aggregates of epidote and quartz are very frequent. Many of the plagioclase crystals seem to be partially changed to epidote.

Although no prehnite was observed in these specimens, the whole mode of occurrence seems to point to an alteration of prehnite as the starting-point for the formation of these tertiary products.

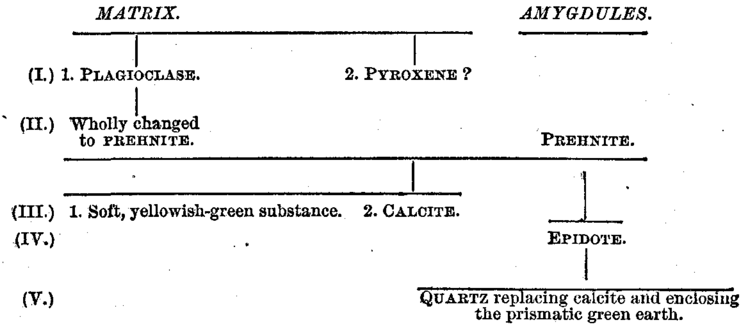
Paragenesis: 1. Pseudo-amygdaloidal chlorite; 2. Epidote and orthoclase; 3. Quartz.

Amygdaloid from Huron Mine. — In the copper-bearing amygdaloid beds of Portage Lake occur frequently considerable masses of amygdaloid, in which the matrix is a coarse, irregular patch-work of massive chloritic substance, of quartz, and of epidote. I have before me a specimen, six inches square, containing all these varieties. One end has a soft, dark-green, chloritic matrix, with large patches of prehnite, more or less altered to soft, light-green substance, and to calcite.

Thin sections show that the matrix has been wholly prehnitized. Fragments of prehnite remain throughout the matrix; but it is mostly changed to a soft, yellowish-green substance.

The amygdules, which were also of prehnite, are now, in places partly, in others wholly, changed. Many of the amygdules consist now partly of calcite, which, from the manner in which it encloses particles of prehnite, and of the soft, yellowish-green product of prehnite, is evidently pseudomorphous after it; the other portion of the same amygdule is often prehnite, more or less changed to a mass of prisms of a light-green substance, which form both on the outside and in the interior of the amygdules. These prisms are monoclinic, and are, probably, epidote. While the calcite encloses the yellowish-green alteration product, it does not contain this epidote-like substance. Where this is in contact with the calcite, the line of separation is sharply drawn. The calcite was formed contemporaneously with the yellowish-green product, or before it; it forms also veinlets through the matrix.

After part of the prehnite had been replaced by calcite, this form of change ended, and the remaining prehnite was subjected to a new process of alteration, — the change to the prismatic substance. In other amygdules a still later phase is apparent; here, after a part of the prehnite had been changed to the yellowish-green substance and calcite, and the rest to the soft, green, prismatic substance, the calcite was replaced by quartz. The quartz forms veinlets in the matrix, cutting those of calcite, while in the amygdules it encloses many fringed-edged fragments of calcite, and also the epidote product of prehnite. The quartz contains many fluid cavities, some with movable bubbles. None of the bubbles showed any change when heated to 100° C.


MATRIX.
AMYGDULES.
(I.) 1. PLAGIOCLASE.
2. PYROXENE ?
(II.) Wholly changed to PREHNITE.
PREHNITE.
(III.) 1. Soft, yellowish-green substance.
2. CALCITE.
(IV.)
EPIDOTE.
(V.)
QUARTZ replacing calcite and enclosing the prismatic green earth.

The chloritic matrix passes gradually into the quartz matrix, which forms the middle zone of the specimen. This consists of purple-brown compact quartz, in which no crystal-faces occur. This portion breaks with sub-conchoidal fracture. It abounds in amygdules of white quartz and green epidote, the latter sometimes in crystals $\frac{1}{10}$ inch long, imbedded in quartz. The amygdules are, in places, half quartz and half calcite.

In thin sections the matrix shows the usual feldspar outlines, filled with a limpid colorless substance, and surrounded and sharply defined by a soft, impellucid, iron-stained substance. Some of the characteristic pseudomorphs after pyroxene also occur.

But in polarized light the bulk of the rock is seen to be quartz, with very coarse granular aggregate polarization, so coarse that each integral grain encloses and fills many feldspar outlines.

In parts of the sections this office is performed by prehnite in the same manner as by quartz. But this is evidently only a fragmentary residue. We have here on a large scale, in the whole matrix, what we saw in the other part of the specimen only in amygdules; here the formerly wholly prehnitized matrix has been almost entirely changed to quartz, perhaps after going through an intermediate change to calcite. Here, too, the quartz encloses large numbers of the delicate prisms of epidote-like substance.

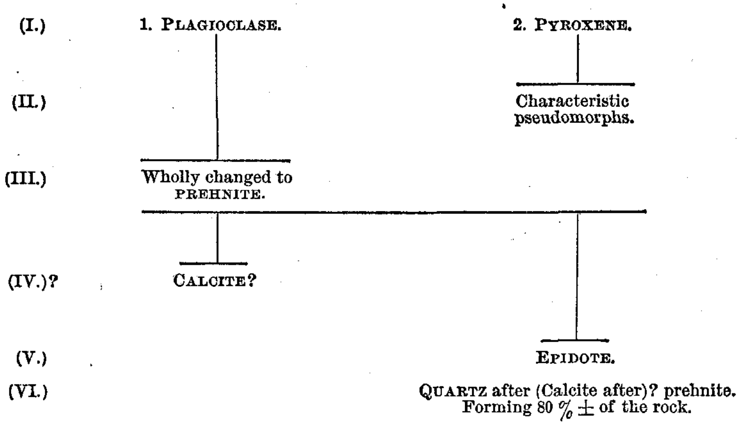
An analysis of this part of the specimen, by Mr. Geo. W. Hawes, gave, as a mean of two analyses, —

Silica	51.42.
Alumina	13.38.
Ferric oxide	9.59.

Ferrous oxide	2.45.
Manganous oxide	.48.
Lime	17.45.
Magnesia	2.14.
Soda	.19.
Water	1.13.
Titanic acid	1.72.
	99.91.

Free quartz, 10.86 per cent. Sp. gr. 3.45.

Metallic copper, 3.43 per cent. Sp. gr., calculated after deducting for quartz and copper, 3.32.

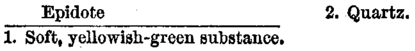

(I.)
1. Plagioclase.
2. Pyroxene.
(II.)
Characteristic pseudomorphs.
(III.)
Wholly changed to PREHNITE.
(IV.)?
Calcite?
(V.)
Epidote.
(VI.)
Quartz after (Calcite after)? prehnite. Forming 80 % ± of the rock.

The rest of the specimen is a very irregular mixture of the purple quartz matrix (with its quartz amygdules), and light-green epidote impregnated with rough threads of copper.

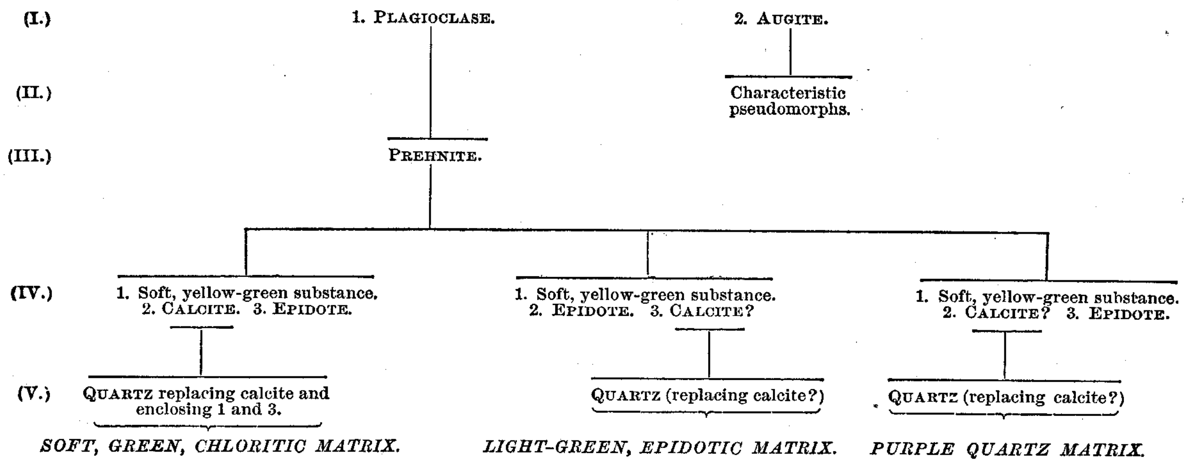
In thin sections it is found to consist of epidote, with more or less distinct crystal outlines, a soft, yellowish-green substance, and quartz which encloses the two other constituents, and permeates the mass in such a manner as to make it evident that it has replaced some pre-existing substance — calcite? — in which the epidote and green substance lay.

That the quartz is younger than the epidote is shown by the fact that crystals of this are included in integral quartz grains.

Paragenesis: —


Epidote
1. Soft, yellowish-green substance.
2. Quartz.

PARAGENESIS OF THE SPECIMEN AS A WHOLE.


(I.)
1. PLAGIOCLASE.
2. AUGITE.
(II.)
Characteristic pseudomorphs.
(III.)
PREHNITE.
(IV.)
1. Soft, yellow-green substance. 2. CALCITE. 3. EPIDOTE.
1. Soft, yellow-green substance. 2. EPIDOTE. 3. CALCITE?
1. Soft, yellow-green substance. 2. CALCITE? 3. EPIDOTE.
(V.)
QUARTZ replacing calcite and enclosing 1 and 3.
QUARTZ (replacing calcite?)
QUARTZ (replacing calcite?)
SOFT, GREEN, CHLORITIC MATRIX.
LIGHT-GREEN, EPIDOTIC MATRIX.
PURPLE QUARTZ MATRIX.

"*Epidote Lode.*" *Mesnard.* — Among the rocks resulting from the alteration of the amygdaloids is one which is generally called epidote, and which seems to have given its name to the "Epidote lode," north of Portage Lake. It is a light-green, hard rock, with an amorphous-looking matrix, and containing amygdules and seams of quartz. Even the naked eye detects countless minute seams of quartz, which traverse the matrix in a manner that suggests the former existence of shrinkage-cracks. In thin sections this rock is seen to consist of bristling masses of a faintly green mineral, in long, thin prisms. These crystals, when seen in very thin parts of the section, under Hartnack's No. 7, and between crossed nicols, generally become dark when oblique to a nicol plane; but occasionally one is found in which the maximum extinction occurs when the longer direction of the prism is parallel to a nicol plane; it is, therefore, a monoclinic mineral.

These crystals are suspended in quartz, which forms a considerable percentage of the rock, not only as amygdules, but as veinlets and matrix.

The quartz is undoubtedly the younger mineral, and the appearance suggests that the rock has been an amygdaloid, of which the matrix was altered finally to this green mineral, and then impregnated with quartz. The resemblance to the Huron mine occurrence suggests that the matrix passed through the prehnite stage.

The absence of free iron-oxide in these epidote-quartz rocks can be explained only by supposing that it entered into combination to form the ferric silicate.

The following is the mean of two analyses made for me, by Mr. G. W. Hawes, in the Laboratory of the Sheffield Scientific School: —

SiO_2	58.87
Al_2O_3	13.05
Fe_2O_3	4.29
FeO	2.70
MnO	.20
CaO	14.88
MgO	1.60
K_2O	.06
Na_2O	.24
H_2O	4.38
	100.27

The free quartz was determined by M. Müller's method,[*] digesting the powder with phosphoric acid. The appearance under the microscope would suggest that much more of the silica should be credited to free quartz than the 10.89 per cent given by Müller's method. I should say that twenty-five to thirty per cent would be nearer the truth. The substance is, probably, quartz, and an epidote very poor in iron.

Sp. gr. 2.88. Free quartz, 10.89.
Sp. gr., after deducting the quartz, 2.91.

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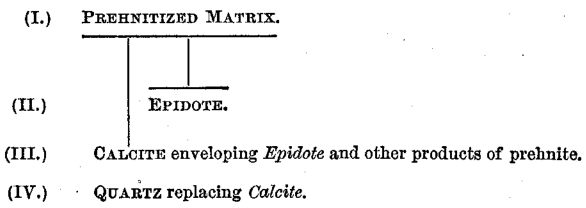
[*] Journ. f. pr. Chemie, xcv. 43, and xcviii. 14.

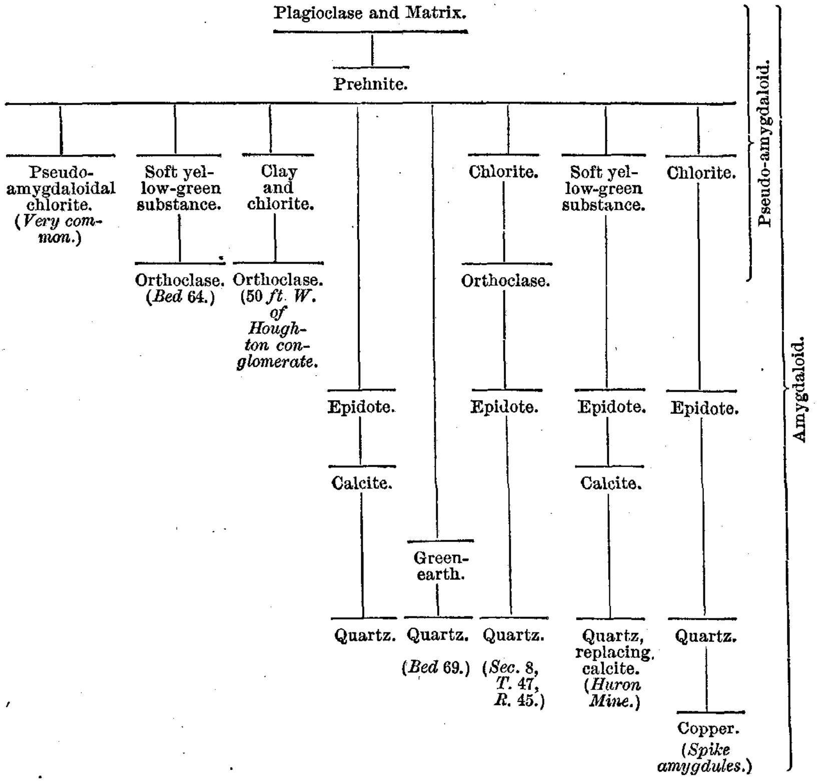
Epidotic Amygdaloid. — In the copper-bearing beds worked at Portage Lake, there occur frequently considerable masses of a hard, light-green, generally fine-grained, epidotic rock, which is often quite free from amygdules. Thin sections show that it consists of a granular aggregate of epidote and quartz, with metallic copper. In parts of the section, the epidote predominates, and the quartz is only just apparent. In other parts, the quartz is in the majority. The grains forming the quartz aggregate are small, and not detrital; they are united without interstices, and were undoubtedly crystallized in the positions they now occupy. The quartz also forms veins, cutting through the more epidotic portions, and throughout the sections it is clearly younger than the epidote; for grains and crystals of the latter are included in the interior of quartz individuals. It is evident that either the two minerals crystallized together, or that the epidote was originally held together by some substance which has been replaced by quartz. The observations on thin sections of the quartz-epidote rock from the Huron justify us in supposing that here, as there, the quartz has replaced some more soluble mineral. On one part of that specimen we saw remnants of prehnite, partially changed to calcite, and also half-finished pseudomorphs of quartz after the same calcite. In another part of that specimen the calcite was gone: but the quartz forming nearly all of the matrix enclosed the same alteration products of prehnite that we saw in the calcite in the other part; while in the epidotic part of the specimen we had simply epidote added as an older constituent, enveloped by quartz.

All my observations seem to indicate clearly, that the epidotic portions of the altered amygdaloids are a product of the alteration of prehnitized matrix.

Its paragenetic relations are, probably, nearly as follows: —


(I.) PREHNITIZED MATRIX.
(II.) EPIDOTE.
(III.) CALCITE enveloping *Epidote* and other products of prehnite.
(IV.) QUARTZ replacing *Calcite*.

I have attempted to correlate the typical ones among the different sets of paragenetic schemes in the annexed table, and to show, by this, which portions of the progressive changes caused the pseudo-amygdaloids or middle zone, and which the more highly altered forms of the amygdaloids proper:—


Plagioclase and Matrix.
Prehnite.
Pseudo-amygdaloidal chlorite. (*Very common.*)
Soft yellow-green substance.
Orthoclase. (*Bed* 64.)
Clay and chlorite.
Orthoclase. (50 *ft. W. of Houghton conglomerate.*)
Epidote.
Calcite.
Quartz.
Green-earth.
Quartz. (*Bed* 69.)
Chlorite.
Orthoclase.
Epidote.
Quartz. (*Sec.* 8, *T.* 47, *R.* 45.)
Soft yellow-green substance.
Epidote.
Calcite.
Quartz, replacing calcite. (*Huron Mine.*)
Chlorite.
Epidote.
Quartz.
Copper. (*Spike amygdules.*)
Pseudo-amygdaloid.
Amygdaloid.

With few exceptions, the course has been as follows:—

Pseudo-amygdaloid stage.

I. Hydration of *chrysolite*, when present.
II. Change of *augite*, loss of lime, and partial loss of iron and magnesia.
III. Change of *feldspar* to *prehnite*, and formation of *prehnite pseudo-amygdules.*
IV. Change of *prehnite* to *chlorite.*
IV *a.* Change of *prehnite* to *orthoclase.*

Amygdaloids.

I. Filling amygdaloidal cavities with *prehnite*, or other minerals. Change of *matrix* to *ferruginous prehnite.*
II. Change of the *prehnite*, in places, to *chlorite;* in others, to *calcite* and *green-earth;* in others, to *epidote* and *calcite.*
III. Entrance of *quartz*, filling all the interstices and replacing the *calcite.*

This is the broader history. Orthoclase is here, as in the pseudo-amygdaloid, of sporadic occurrence, and a product of the prehnite.

The changes under II. may affect only the amygdules, or, if the matrix was prehnitized, it applies to the whole mass of the amygdaloid. It does this in such a manner that, where carried to its extremes, considerable portions of the bed have lost every semblance of an amygdaloid, and consist now of chlorite, epidote, calcite, and quartz, more or less intimately associated, or forming larger masses, of the most indefinite shapes, and merging into each other. Sometimes portions of partially altered prehnite occur. In places, considerable masses of rich brown, and green fresh prehnite filled with copper occur; but, as a rule, this mineral has given way to its products.

To this process, the copper-bearing beds of Portage Lake — wrongly called lodes — owe their origin. Considerable portions of these beds are but partially altered amygdaloids, containing amygdules of prehnite, chlorite, calcite, or quartz, with more or less copper; other portions are in the condition described above.

This, too, (II. and III.), appears to have been the principal period of concentration of the copper. In the still amygdaloidal portions, this metal was deposited in the cavities and in cleavage-planes of some minerals, and replaced calcite amygdules, &c. But in the confused and highly altered parts of the bed it crystallized free, where it had a chance: more generally it replaced other minerals on a considerable scale. It formed, in calcite bodies, those irregular, solid, branching forms, that are locally known as horn-copper, often many hundred pounds in weight; in the epidote, quartz, and prehnite bodies, it occurs as thread and flake-like impregnations; in the foliaceous lenticular chloritic bodies, it formed flakes between the cleavage planes and oblique joints, or in places — and this is more particularly true of the fissure-veins, which we are not now considering — it replaces the chloritic, selvage-like substance till it forms literally pseudomorphs, sometimes several hundred tons in weight.

When the amygdaloid has arrived at the condition we have been describing, it assumes some of the characters of a vein, in that, although it presents no open fissure, it contains greater or smaller masses of calcite and other minerals that are easily replaced by an intruder. To this period, probably, belongs the replacement of calcite by datolite; and here, also, the rather rare occurrence of analcite crystals, and the pseudomorphs of orthoclase after these.

As I have already remarked, the pseudo-amygdaloids are merely altered forms of the same rock as the lower zone. There seems to be a definite limit at which this progressive change stops, and that is when all the augite is changed to its green pseudomorph, and a large percentage of the rest of the rock consists of pseudo-amygdules of delessite, and partial pseudomorphs of this after plagioclase. The occurrence of epidote and quartz is not general, and is then confined to scattering pseudo-amygdules, in which these minerals have succeeded prehnite, perhaps in the local absence of the conditions necessary to produce the usual delessite.

Thus I conceive that the extent of the change to the pseudo-amygdaloidal form is conditioned essentially by the amount of augite present, to supply first the lime necessary to aid in changing the plagioclase to prehnite, and next the iron and magnesia to form the delessite, whether by acting directly on the feldspar substance or on the prehnite.

The amygdaloids proper were, probably, both structurally and chemically, somewhat different from the lower zone, in that it is reasonable to suppose that, in addition to being more or less porous, they contained a greater or less amount of amorphous base, which is more easily altered than a crystalline aggregate. But, from whatever cause, the amygdaloids have, as we have seen, been capable of much greater changes than the lower zone: in them the tendency is undoubtedly towards the formation of quartz, chlorite, and epidote rocks as a more stable limit, through the mediation of prehnite and calcite.

There is one form of change in some amygdaloids that I have not mentioned; namely, the laumontitic. I have not had sufficiently good material to work upon, and have not studied it to any extent under the microscope. Wherever I have been able to determine its relative age, as compared with prehnite, I have found it to be older than the latter. The macroscopic appearance of some beds of this amygdaloid suggest, that they have not only had the cavities filled with laumontite, but that the matrix has been more or less changed to the same zeolite.

XXI.

INVESTIGATIONS IN QUATERNIONS.

THE THESIS OF A CANDIDATE FOR MATHEMATICAL HONORS CONFERRED WITH THE DEGREE OF A.B., AT HARVARD COLLEGE, AT COMMENCEMENT, 1877.

By Washington Irving Stringham.

Presented by Professor Benjamin Peirce, Jan. 9, 1878.

I. Logarithms of Quaternions.

1°. Hamilton proves (Lectures, p. 546) the following theorem: The tensor of the sum of any number of quaternions cannot exceed the sum of their tensors.

If p and q be any two quaternions, then

$$\begin{aligned} \mathrm{T}(p+q)^2 &= (p+q)(\mathrm{K}p+\mathrm{K}q) = \mathrm{T}p^2 + \mathrm{T}q^2 + 2\mathrm{S}.p\mathrm{K}q \\ &= \mathrm{T}p^2 + \mathrm{T}q^2 + 2\mathrm{T}p\mathrm{T}q\mathrm{SU}.p\mathrm{K}q \\ &= (\mathrm{T}p + \mathrm{T}q)^2 - 2\mathrm{T}p\mathrm{T}q(1 - \mathrm{SU}.p\mathrm{K}q)\,; \end{aligned}$$

and since the scalar of a versor cannot fall outside the limits ± 1,

$$\therefore\ \mathrm{T}(p+q) \mathrel{-\!\!<} \mathrm{T}p + \mathrm{T}q.$$

The same proof holds for any number of quaternions, and gives in general

$$\mathrm{T}(p+q+r+\ldots) \mathrel{-\!\!<} \mathrm{T}p + \mathrm{T}q + \mathrm{T}r + \ldots$$

On the foregoing theorem depends the test of convergence of the series

$$\Sigma = 1 + q + \frac{q^2}{2\,!} + \frac{q^3}{3\,!} + \ldots$$

Let Σ_n represent the first n terms of Σ; $S =$ the series formed by the tensors of Σ; $S_n =$ the first n terms of S. The terms of S are represented by the general form $\frac{\mathrm{T}q^n}{n\,!} = a_n$. Each successive term of this series is formed by multiplying the preceding term by a factor of the form $\frac{\mathrm{T}q}{n}$. But n continually increases, while $\mathrm{T}q$ remains constant. Hence, however great a finite value $\mathrm{T}q$ may have, n will finally become greater than $\mathrm{T}q$, and still remain finite. It will therefore be possible to take n so great, and still finite, that the ratio between the last two terms of S_n shall be finitely greater than unity. Insert k, a finite quantity, between this limit and unity. Then

$$\frac{a_n}{a_{n+1}} > k > 1, \qquad \text{whence}$$

$a_{n+1} < \frac{a_n}{k}$, $a_{n+2} < \frac{a_{n+1}}{k} < \frac{a_n}{k^2}$, and in general $a_{n+p} < \frac{a_n}{k^p}$. Then, designating by $S_{n,p}$ the p terms from a_n to a_{n+p-1} inclusive, we have

$$S_{n,p} = a_n + a_{n+1} + \ldots + a_{n+p-1} < a_n \left(1 + \frac{1}{k} + \ldots + \frac{1}{k^{p-1}}\right)$$

or

$$S_{n,p} < a_n \left(\frac{1}{1-\frac{1}{k}} - \frac{\frac{1}{k^p}}{1-\frac{1}{k}}\right), \quad S_{n,p} < a_n \left(\frac{k}{k-1} - \frac{k}{k^p(k-1)}\right).$$

When $p = \infty$, $S_{n,p} < a_n \frac{k}{k-1}$, and therefore $S_{n,\infty}$ is finite, and since S_n is finite, therefore $S_n + S_{n,\infty} = S$ is finite, and S is convergent. But $\mathrm{T}\Sigma \mathrel{-\!\!<} S$ and $\therefore$ $\mathrm{T}\Sigma$ is finite. If the tensor of a quaternion is finite, the quaternion is finite; and hence, as was to be proved, the series

$$\Sigma = 1 + q + \frac{q^2}{2\,!} + \ldots$$

is convergent. This series is the exponential of quaternions, and may be designated by the usual symbol ϵ^q. It is a quaternion complanar with q.

2°. In ordinary scalar analysis, the fundamental principle of the exponential function consists in the relation

$$\epsilon^{q+q'} = \epsilon^q \epsilon^{q'},$$

where q and q' are numbers. In quaternion analysis, this principle holds good only for complanar quaternions, in which the commutative property always obtains. The proof I give is precisely similar to that given for complex scalar quantities by MM. Briot et Bouquet, Fonc. Ellipt. p. 89.

Let Σ_n, Σ'_n, Σ''_n, represent the sums of the first n terms of the series which define the functions ϵ^{q}, $\epsilon^{q'}$, $\epsilon^{q+q'}$: —

$$\Sigma_n = 1 + q + \frac{q^2}{2\,!} + \dots \frac{q^{n-1}}{(n-1)\,!},$$

$$\Sigma'_n = 1 + q' + \frac{q'^2}{2\,!} + \dots \frac{q'^{n-1}}{(n-1)\,!},$$

$$\Sigma''_n = 1 + (q + q') + \dots \frac{(q+q')^{n-1}}{(n-1)\,!}.$$

Also, represent the functions ϵ^{Tq}, $\epsilon^{Tq'}$, $\epsilon^{Tq+Tq'}$, down to n terms, by

$$S_n = 1 + Tq + \frac{Tq^2}{2\,!} + \dots \frac{Tq^{n-1}}{(n-1)\,!},$$

$$S'_n = 1 + Tq' + \frac{Tq'^2}{2\,!} + \dots \frac{Tq'^{n-1}}{(n-1)\,!},$$

$$S''_n = 1 + (Tq + Tq') + \dots \frac{(Tq + Tq')^{n-1}}{(n-1)\,!}.$$

Of this last series, take the sum of the first $(2n - 1)$ terms: —

$$S'_{2n-1} = 1 + (Tq + Tq') + \dots + \frac{(Tq + Tq')^{2n-2}}{(2n-2)\,!}.$$

By developing the numerators of S''_n and S''_{2n-1}, and effecting the product $S_nS'_n$, we shall discover (since the multiplications are commutative) that all the terms of S''_n are to be found in the product $S_nS'_n$, and that all the terms of this product are completely represented by corresponding terms in S''_{2n-1}. Hence

$$S'_n < S_n'S_n < S''_{2n-1}.$$

When n increases indefinitely, S''_n and S''_{2n-1} tend to the same limit $\epsilon^{Tq+Tq'}$;

$$\therefore \lim_{n=\infty} (S_nS'_n - S''_n) = 0.$$

The property, that the tensor of the sum of any number of quaternions is as small as the sum of the tensors, together with the fact that the terms of the expression $(\Sigma_n\Sigma'_n - \Sigma''_n)$ have for their tensors the corresponding terms of $(S_nS'_n - S''_n)$, gives

$$T(\Sigma_n\Sigma'_n - \Sigma''_n) -\!\!< S_nS'_n - S''_n,$$

and

$$\therefore \lim_{n=\infty} T(\Sigma_n\Sigma'_n - \Sigma''_n) = 0.$$

Now when Tq is null, q itself is null, and

$$\therefore \lim (\Sigma_n \Sigma'_n - \Sigma''_n) = 0.$$

Thus at the limit $\Sigma_n \Sigma'_n = \Sigma''_n,$

or $$\epsilon^q \epsilon^{q'} = \epsilon^{q + q'}. \tag{1}$$

This proof includes the proof of $\epsilon^q = \epsilon^{\mathrm{S}q} \epsilon^{\mathrm{V}q}$, since in the products of scalars with vectors the commutative property holds.

With diplanar quaternions, formula (1) is not true, since such quaternions are not commutative with each other.

3°. Defining the five quaternion functions ϵ^q, sin q, cos q, Sh q, Ch q, by the five series

$$(a) \quad \epsilon^q = \sum_0^\infty \frac{q^n}{n\,!} = 1 + q + \frac{q^2}{2\,!} + \frac{q^3}{3\,!} + \ldots$$

$$(b) \quad \sin q = \sum_0^\infty (-)^n \frac{q^{2n+1}}{(2n+1)\,!} = q - \frac{q^3}{3\,!} + \frac{q^5}{5\,!} - \ldots$$

$$(c) \quad \cos q = \sum_0^\infty (-)^n \frac{q^{2n}}{(2n)\,!} = 1 - \frac{q^2}{2\,!} + \frac{q^4}{4\,!} - \ldots$$

$$(d) \quad \mathrm{Sh}\ q = \sum_0^\infty \frac{q^{2n+1}}{(2n+1)\,!} = q + \frac{q^3}{3\,!} + \frac{q^5}{5\,!} + \ldots$$

$$(e) \quad \mathrm{Ch}\ q = \sum_0^\infty \frac{q^{2n}}{(2n)\,!} = 1 + \frac{q^2}{2\,!} + \frac{q^4}{4\,!} + \ldots$$

we have, by adding (*d*) and (*e*)

$$\epsilon^q = \mathrm{Ch}\ q + \mathrm{Sh}\ q, \tag{2}$$

and by taking for q the particular value Vq,

$$\epsilon^{\mathrm{V}q} = \mathrm{ChV}q + \mathrm{ShV}q;$$

whence observing that $\epsilon^q = \epsilon^{\mathrm{S}q} \epsilon^{\mathrm{V}q}$, we obtain the important general formula

$$\epsilon^q = \epsilon^{\mathrm{S}q} (\mathrm{ChV}q + \mathrm{ShV}q). \tag{3}$$

If $\mathfrak{i}$ denotes any value of $\sqrt{-1}$ which is commutative with q, formulae (*b*), (*c*), (*d*), and (*e*) give

$$\sin q\mathfrak{i} = \mathfrak{i}\,\mathrm{Sh}\,q, \qquad \cos q\mathfrak{i} = \mathrm{Ch}\,q,$$

$$\mathrm{Sh}\,q\mathfrak{i} = \mathfrak{i}\sin q, \qquad \mathrm{Ch}\,q\mathfrak{i} = \cos q;$$

and (2) gives

$$\epsilon^{q\mathfrak{i}} = \cos q + \mathfrak{i}\sin q. \qquad (4)$$

Thus, taking $\mathrm{TV}q$ for q, and $\mathrm{UV}q$ for $\mathfrak{i}$, we can write formula (3) as follows: —

$$\epsilon^{q} = \epsilon^{\mathrm{S}q}\,(\cos \mathrm{TV}q + \mathrm{UV}q\,.\sin \mathrm{TV}q). \qquad (5)$$

In the special case of a vector, (3) and (5) become

$$\epsilon^{\alpha} = \mathrm{Ch}\,\alpha + \mathrm{Sh}\,\alpha \qquad (3)'$$

$$\epsilon^{\alpha} = \cos \mathrm{T}\alpha + \mathrm{U}\alpha \sin \mathrm{T}\alpha, \qquad (5)'$$

where α is any vector whatever.

4°. Formulæ, analogous to those just given, may be deduced for biquaternions.

Let $\mathrm{Q} = q_1 + \mathfrak{i}q_2$, where $\mathfrak{i} = \sqrt{-1}$ (a scalar), and q_1 and q_2 are complanar. We have $\mathrm{UV}q_2 = \pm\,\mathrm{UV}q_1$. Since $\mathfrak{i}$ is commutative with any quaternion, we have, by (4), —

$$\epsilon^{\mathfrak{i}\mathrm{V}q_2} = (\cos \mathrm{V}q_2 + \mathfrak{i}\sin \mathrm{V}q_2),$$

which, multiplied by (5) and by $\epsilon^{\mathfrak{i}\mathrm{S}q_2}$, gives

$$\epsilon^{q_1+\mathfrak{i}q_2} = \epsilon^{\mathrm{S}(q_1+\mathfrak{i}q_2)}(\cos \mathrm{TV}q_1 + \mathrm{UV}q_1 \sin \mathrm{TV}q_1)(\cos \mathrm{V}q_2 + \mathfrak{i}\sin \mathrm{V}q_2)$$

$$= \epsilon^{\mathrm{SQ}}(\cos \mathrm{TV}q_1 + \mathrm{UV}q_1 \sin \mathrm{TV}q_1)(\mathrm{Ch}\,\mathrm{TV}q_2 + \mathfrak{i}\mathrm{UV}q_2\,\mathrm{Sh}\,\mathrm{TV}q_2)$$

$$= \epsilon^{\mathrm{SQ}}\left[\begin{array}{l}\cos \mathrm{TV}q_1\,\mathrm{Ch}\,\mathrm{TV}q_2 \pm \mathfrak{i}\mathrm{UV}q_1 \cos \mathrm{TV}q_1\,\mathrm{Sh}\,\mathrm{TV}q_2 \\ \mp\,\mathfrak{i}\sin \mathrm{TV}q_1\,\mathrm{Sh}\,\mathrm{TV}q_2 + \mathrm{UV}q_1 \sin \mathrm{TV}q_1\,\mathrm{Ch}\,\mathrm{TV}q_2\end{array}\right];$$

$$\epsilon^{\mathrm{Q}} = \epsilon^{\mathrm{SQ}}[\cos(\mathrm{TV}q_1 \pm \mathfrak{i}\mathrm{TV}q_2) + \mathrm{UV}q_1 \sin(\mathrm{TV}q_1 \pm \mathfrak{i}\mathrm{TV}q_2)]. \qquad (6)$$

Observing that

$$\cos(\mathrm{TV}q_1 \pm \mathfrak{i}\mathrm{TV}q_2) = \mathrm{Ch}\,\mathrm{V}(q_1 + \mathfrak{i}q_2),$$

and that

$$\sin(\mathrm{TV}q_1 \pm \mathfrak{i}\mathrm{TV}q_2) = -\,\mathrm{UV}q_1\,\mathrm{Sh}\,\mathrm{V}(q_1 + \mathfrak{i}q_2),$$

and substituting these values in (6), we obtain this other important formula

$$\epsilon^{\mathrm{Q}} = \epsilon^{\mathrm{SQ}}(\mathrm{Ch}\,\mathrm{VQ} + \mathrm{Sh}\,\mathrm{VQ}). \qquad (7)$$

For bivectors, we have, if $\alpha = \alpha_1 + \mathrm{i}\alpha_2$, provided that α_1 and α_2 are parallel,

$$\epsilon^{\alpha} = \cos\,(\mathrm{T}\alpha_1 + \mathrm{iT}\alpha_2) + \mathrm{U}\alpha \sin\,(\mathrm{T}\alpha_1 + \mathrm{iT}\alpha_2),$$

$$= \mathrm{Ch}\;\alpha + \mathrm{Sh}\;\alpha.$$

5°. The usual algebraic definition of the natural logarithm, applied to a quaternion, gives

$$\epsilon^{\log q} = q. \qquad (8)$$

Let $$\log q = p, \log q' = p'.$$

$$\therefore\; \epsilon^{p} = q,\; \epsilon^{p'} = q',$$

and by (1) $$\epsilon^{p+p'} = qq',$$

which gives, for complanar (but not for diplanar) quaternions, the general formula

$$\log qq' = \log q + \log q'. \qquad (9)$$

This formula, however, is subject to certain limitations as will be shown later, in 9°. But in particular and always, we have

$$\log q = \log \mathrm{T}q\mathrm{U}q = \log \mathrm{T}q + \log \mathrm{U}q. \qquad (10)$$

6°. Let $\log\mathrm{U}q = q$, where q' is an undetermined quaternion. By (5)

$$\epsilon^{q'} = \mathrm{U}q = \epsilon^{\mathrm{S}q'}(\cos \mathrm{TV}q' + \mathrm{UV}q' \sin \mathrm{TV}q'),$$

whence,

$$\mathrm{SU}q = \epsilon^{\mathrm{S}q'} \cos \mathrm{TV}q' = \cos \angle q,$$

$$\mathrm{TVU}q = \mathrm{T}.\;\epsilon^{\mathrm{S}q'} \sin \mathrm{TV}q' = \sin \angle q\,;$$

$$\therefore\; \mathrm{T}^2\mathrm{VU}q + \mathrm{S}^2\mathrm{U}q' = \epsilon^{2\mathrm{S}q'} = 1,$$

$$\therefore\; \epsilon^{\mathrm{S}q'} = (-)^n\, 1, \text{ and } \mathrm{S}q' = n\,\varpi\,\mathrm{i};$$

where i is the scalar $\sqrt{-1}$, and n any integer. Comparing these results, we find

$$\cos \angle q = (-)^n \cos \mathrm{TV}q',\; \sin \angle q = \mathrm{T} \sin \mathrm{TV}q',$$

$$\pm\, \mathrm{TV}q' = \angle q + k\,\varpi, \qquad \mathrm{V}q' = \pm\,(\angle q + k\,\varpi)\mathrm{UV}q',$$

where k is any integer which differs from n by an even number. But

$$\pm\, \mathrm{UV}q' = \mathrm{UVU}q, \text{ and } \mathrm{V}q \parallel \mathrm{VU}q,$$

and therefore

$$Vq' = (\angle q + k\,\text{ə})UVq,$$

and since

$$Sq' = n\,\text{ə}\,i,$$

$$\therefore\ q' = n\,\text{ə}\,i + (\angle q + k\,\text{ə})UVq,$$

whence follows the fundamental formula

$$\log q = \log Tq + n\,\text{ə}\,i + (\angle q + k\,\text{ə})UVq; \qquad (11)$$

which can be more accurately written

$$\log q = \log Tq + n\,\text{ə}\,i + [\angle q + (n + 2k)\,\text{ə}]UVq. \qquad (12)$$

If q be negative, this becomes

$$\log(-q) = \log Tq + n\text{ə}i + [\angle q + (n + 2k + 1)\text{ə}]UVq. \quad (13)$$

If q be a vector, its angle is $\frac{\text{ə}}{2}$ and

$$\log(\pm\,\alpha) = \log T\alpha + n\,\text{ə}\,i \pm \frac{2(n + 2k) + 1}{2}\,\text{ə}U\alpha. \quad (14)$$

If $n = k = 0$, and if the *real* logarithm of Tq be taken, we have what Hamilton calls the *principal* logarithm, or simply *the Logarithm*, of a quaternion, and indicates by using l instead of log. Thus we have

$$lq = lTq + \angle qUVq. \qquad (12)'$$

The principal logarithm of q is a real quaternion complanar with q itself. Its scalar and vector parts are

$$Slq = lTq,\ Vlq = \angle qUVq.$$

7°. The formulae of 4° render it possible to find the logarithm of a biquaternion. Let

$$\log UQ = Q' = q'_1 + iq'_2,$$

where Q' is an undetermined biquaternion, and $Q = q_1 + iq_2$ is such that $UVq_1 = UVq_2$. Formula (6) gives

$$SUQ = \text{ϭ}^{SQ'} \cos(TVq'_1 + iTVq'_2),$$

$$TVUQ = T\text{ϭ}^{SQ'} \sin(TVq'_1 + iTVq'_2),$$

$$\therefore\ T^2VUQ + S^2UQ = \text{ϭ}^{2SQ'} = 1. \qquad (15)$$

Let us agree to define the angle of Q as such that

$$\mathrm{SUQ} = \cos \angle \mathrm{Q}, \text{ and } \mathrm{TVUQ} = \sin \angle \mathrm{Q}.$$

This definition will be admissible if it be not found inconsistent with other established definitions. Then (15) gives

$$\mathrm{SQ}' = n\,\text{ə}\,\mathrm{i},$$

and

$$\angle \mathrm{Q} = \mathrm{TV}q'_1 + \mathrm{i}\mathrm{TV}q'_2 + k\,\text{ə},$$

n and k being any integers, which differ by an even number. Multiplying $\angle \mathrm{Q}$ into $\mathrm{UV}q_1$, we have

$$\angle \mathrm{Q}.\mathrm{UV}q_1 = \mathrm{V}(q'_1 + \mathrm{i}q'_2) + k\text{ə}\mathrm{UV}q_1,$$

$$(\angle \mathrm{Q} + k\text{ə})\mathrm{UV}q_1 = \mathrm{V}(q'_1 + \mathrm{i}q'_2) = \mathrm{VQ}',$$

or

$$\mathrm{Q}' = n\text{ə}\mathrm{i} + [\angle \mathrm{Q} + (n + 2\,k)\text{ə}]\mathrm{UV}q_1 = \log \mathrm{UQ}.$$

Whence, for the general logarithm of a biquaternion, of which the real components are complanar, we find

$$\begin{aligned} \log \mathrm{Q} &= \log \mathrm{TQ} + \log \mathrm{UQ} \\ &= \log \mathrm{TQ} + n\text{ə}\mathrm{i} + [\angle \mathrm{Q} + (n + 2k)\text{ə}]\mathrm{UV}q_1; \quad (16) \end{aligned}$$

and, for the principal logarithm of such a biquaternion,

$$\mathrm{l}\,\mathrm{Q} = \mathrm{l}\,\mathrm{TQ} + (\angle \mathrm{Q})\mathrm{UV}q_1. \quad (17)$$

8°. For the product of any number of complanar quaternions (when the sum of their angles does not exceed 180°) we have by (9)

$$\log (q.\, q'.\, q''.\, \ldots) = \log q + \log q' + \ldots,$$

or

$$\log \Pi q = \Sigma \log q; \quad (18)$$

and since $\mathrm{UV}q = \mathrm{UV}q' = \mathrm{UV}q''$ etc., we therefore obtain at once from (12) the general formula

$$\begin{aligned} \log \Pi q &= \Sigma \log \mathrm{T}q + n\text{ə}\mathrm{i} + [\Sigma\angle q + (n + 2k)\text{ə}]\mathrm{UV}q \\ &= \log \Pi\mathrm{T}q + n\text{ə}\mathrm{i} + [\angle \Pi q + (n + 2k)\text{ə}]\mathrm{UV}q. \quad (19) \end{aligned}$$

This formula gives the principal logarithm only when $(\Sigma\angle q) < 180°$ (9°). The two expressions of this formula show that in general ($\Sigma\angle q$ being $< 180°$),

$$\angle \Pi q = \Sigma\angle q;$$

which is otherwise easily seen to be true.

9°. The angle of a quaternion has not the usual generality of an angle in trigonometry; it is never negative or greater than 180°. (19) must, then, be modified as follows, to give the principal logarithm. It is easily seen that if

$$(\Sigma\angle q) > 2m\text{ə} \text{ and } < (2m+1)\text{ə},$$

$$\angle \Pi q = \Sigma\angle q - 2m\text{ə},$$

and that, if

$$(\Sigma\angle q) > (2m+1)\text{ə} \text{ and } < (2m+2)\text{ə},$$

$$\angle \Pi q = (2m+2)\text{ə} - \Sigma\angle q.$$

Hence, the principal logarithm of Πq becomes

$$\mathrm{l}\Pi q = \Sigma\,\mathrm{lT}q + (\Sigma\angle q - 2m\text{ə})\mathrm{UV}q, \tag{20}$$

$$\text{if } (\Sigma\angle q) > 2m\text{ə} \text{ and } < (2m+1)\text{ə};$$

or

$$\mathrm{l}\Pi q = \Sigma\,\mathrm{lT}q + [\Sigma\angle q - (2m+2)\text{ə}]\mathrm{UV}q, \tag{20$'$}$$

$$\text{if } (\Sigma\angle q) > (2m+1)\text{ə} \text{ and } < (2m+2)\text{ə}.$$

For the power of a quaternion, these formulae become

$$\mathrm{l}q^n = n\mathrm{lT}q + (n\angle q - 2m\text{ə})\mathrm{UV}q, \tag{21}$$

$$\text{if } (n\angle q) > 2m\text{ə} \text{ and } < (2m+1)\text{ə};$$

or

$$\mathrm{l}q^n = n\mathrm{lT}q + [n\angle q - (2m+2)\text{ə}]\mathrm{UV}q, \tag{21$'$}$$

$$\text{if } (n\angle q) > (2m+1)\text{ə} \text{ and } < (2m+2)\text{ə}.$$

For the principal logarithm of any integral power of a vector: —

$$\mathrm{l}\alpha^{4n} = 4n\mathrm{lT}\alpha,$$

$$\mathrm{l}\alpha^{4n+1} = (4n+1)\mathrm{lT}\alpha + \tfrac{1}{2}\text{ə}\mathrm{U}\alpha,$$

$$\mathrm{l}\alpha^{4n+2} = (4n+2)\mathrm{lT}\alpha \pm \text{ə}\mathrm{U}\alpha,^*$$

$$\mathrm{l}\alpha^{4n+3} = (4n+3)\mathrm{lT}\alpha - \tfrac{1}{2}\text{ə}\mathrm{U}\alpha.$$

The cases in which $\mathrm{l}q^n = n\mathrm{l}q$ are those in which $(n\angle q) < \text{ə}$.

For $\mathrm{l}q^{\frac{1}{n}}$ we shall always have

$$\mathrm{l}q^{\frac{1}{n}} = \tfrac{1}{n}\mathrm{lT}q + \tfrac{1}{n}\angle q.\mathrm{UV}q,$$

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* But for **U**α in this formula, *any* unit-vector may be substituted.

since $(\angle q) < \text{ə}$ and $\therefore\ (\frac{1}{n}\angle q) < \text{ə}$; hence, in general,

$$\mathrm{l}q^{\frac{1}{n}} = \tfrac{1}{n}\mathrm{l}q.$$

Hence $$\mathrm{l}q^{\frac{m}{n}} = \tfrac{m}{n}\mathrm{l}q, \text{ if } (\tfrac{m}{n}\angle q) < \text{ə}.$$

Again $\angle q^{-1} = \angle q$, $\mathrm{UV}q^{-1} = -\mathrm{UV}q$; and hence

$$\mathrm{l}q^{-1} = -\mathrm{lT}q - \angle q.\mathrm{UV}q,$$

$$\mathrm{l}q^{-n} = -n\mathrm{lT}q - n\angle q.\mathrm{UV}q, \text{ if } (n\angle q) < \text{ə}.$$

We see then that, in general, n being integral or fractional, positive or negative, and q being any quaternion,

$$\mathrm{l}q^{n} = n\mathrm{l}q, \text{ if } -\text{ə} < (n\angle q) < +\text{ə}. \tag{22}$$

10°. The following results are deduced immediately from those already obtained: —

$$\mathrm{l}(-q) = \mathrm{lT}q - (\text{ə} - \angle q)\mathrm{UV}q$$

$$= \mathrm{l}q - \text{ə}\mathrm{UV}q = \mathrm{l}q + \text{ə}\mathrm{UV}(-q), \tag{23}$$

$$\mathrm{lK}q = \mathrm{lT}q - \angle q.\mathrm{UV}q = \mathrm{Kl}q, \tag{24}$$

$$\mathrm{l}q^{-1} = -\mathrm{lT}q - \angle q.\mathrm{UV}q = -\mathrm{l}q, \tag{25}$$

$$\mathrm{l}\tfrac{\alpha}{\beta} = \mathrm{lT}\alpha - \mathrm{lT}\beta + \angle\tfrac{\alpha}{\beta}\mathrm{UV}\beta\alpha, \tag{26}$$

$$\mathrm{l}\alpha\beta = \mathrm{lT}\alpha + \mathrm{lT}\beta + (\text{ə} - \angle\tfrac{\beta}{\alpha})\mathrm{UV}\alpha\beta. \tag{27}$$

11°. Suppose a series of diplanar quaternions to be such that

$$\mathrm{U}q = \mathrm{U}\tfrac{\beta}{\alpha},\ \mathrm{U}q' = \mathrm{U}\tfrac{\gamma}{\beta},\ \mathrm{U}q'' = \mathrm{U}\tfrac{\delta}{\gamma}, \text{ etc.,}$$

where $\alpha, \beta, \gamma, \delta, \ldots \omega$, are any vectors whatever. Then

$$\mathrm{U}\delta = \mathrm{U}q''\gamma,\ \mathrm{U}\delta\beta^{-1} = \mathrm{U}q''\gamma\beta^{-1} = \mathrm{U}q''q',$$

$$\mathrm{U}q''q'q = \mathrm{U}\delta\beta^{-1}\beta\alpha^{-1} = \mathrm{U}\delta\alpha^{-1},$$

and, continuing this process, we shall finally have

$$\mathrm{U}(q_{\omega}\ldots q''q'q) = \mathrm{U}\tfrac{\omega}{\alpha},$$

where ω is the last vector of the series. Hence, the product of such a series of quaternions will be

$$q_\omega \dots q''q'q = \mathrm{T}\,(q_\omega \dots q''q'q)\mathrm{U}\frac{\omega}{\alpha}.$$

The principal logarithm of such a product is

$$\mathrm{l}(q_\omega \dots q''q'q) = \Sigma \mathrm{lT}q + \angle{}^{\omega}_{\alpha}\,\mathrm{UV}\frac{\omega}{\alpha},$$

or

$$\mathrm{l}\Pi q = \Sigma \mathrm{lT}q + \angle\,\Pi q.\mathrm{UV}\,\Pi q. \tag{28}$$

The order of the factors must be carefully observed in using this formula.

12°. Let us consider the logarithm of the product of a series of complanar vectors.

First, suppose the number of factors to be even. Pair the successive vectors, and regard these pairs as quaternions. Since they are complanar quaternions, we have, by (20), (20)′, and (27),

$$\mathrm{l}\,(\alpha_1\alpha_2.\alpha_3\alpha_4.\alpha_5\alpha_6 \dots \alpha_{2n-1}\alpha_{2n}) =$$

$$\sum_1^{2n} \mathrm{lT}\alpha_k + \left[(n - 2m)\partial - \sum_1^{n} \angle \frac{\alpha_{2k}}{\alpha_{2k-1}}\right]\mathrm{UV}\alpha_1\alpha_2, \tag{29}$$

where $\quad (n - 2m + 1)\partial > \left(\Sigma \angle \frac{\alpha_{2k}}{\alpha_{2k-1}}\right) > (n - 2m - 1)\partial,$

and where the angles are to be taken positively or negatively according as they agree or disagree with $\frac{\alpha_2}{\alpha_1}$.

Secondly, if the number of factors be odd, then regard their product, down to and excluding the last vector α_{2n}, as the product of a given number of complanar quaternions. This product, it will be observed, is a quaternion q whose plane is the same as that of the vector factors. The remaining odd vector lies in the plane of q; so that $q\alpha_{2n}$ is a vector lying in the plane of q. Therefore the product of any odd number of complanar vectors is a vector complanar with those vectors. Hence, if $\alpha_0\alpha_1\alpha_2 \dots \alpha_{2n}$ be the given vectors,

$$\mathrm{l}\,(\alpha_0\alpha_1\alpha_2 \dots \alpha_{2n}) = \sum_0^{2n} \mathrm{lT}\alpha_k + \frac{\partial}{2}\,\mathrm{U}\prod_0^{2n}\alpha_k. \tag{30}$$

a formula true for any odd number of complanar vectors.

II. APPLICATIONS OF QUATERNION ANALYSIS TO RECTIFICATION OF CURVES, QUADRATURE OF SURFACES, AND CUBATURE OF SOLIDS.

1°. If $\rho = \varphi\ (t)$ be the equation of any curve in space, it is easily shown that the complete derivative of ρ, relatively to the scalar variable involved, is the tangent to the curve; —

$$\mathrm{D}_t\rho = \rho' = \varphi'\ (t). \tag{1}$$

Here $\mathrm{T}\rho'$ is the derivative, and $\mathrm{T}\rho'\mathrm{d}t$ the element, of the arc of the curve. Hence, any length of arc will be

$$s - s_0 = \int_{t_0}^{t} \mathrm{T}\rho'. \tag{2}$$

2°. Again, the element of double area swept by the radius vector will be $\mathrm{TV}\rho\rho'\mathrm{d}t$, and any finite area swept by ρ will be

$$A = \tfrac{1}{2}\int_{t_0}^{t} \mathrm{TV}\rho\rho'. \tag{3}$$

If the surface be plane, we may change the origin of vectors to a point in the plane, and so find an area measured from the new origin and limited by the limiting positions of the new generating vector.

Let δ be the vector of the new origin (from any origin whatever in space). Then the generating vector will have the form

$$\varpi = \rho - \delta,$$

and the finite area will be

$$A = \tfrac{1}{2}\int_{t_0}^{t} \mathrm{TV}\varpi\varpi' = \tfrac{1}{2}\int_{t_0}^{t} \mathrm{TV}(\rho - \delta)\rho'. \tag{4}$$

3°. If a plane curve be revolved about a given axis lying in its plane, the surface generated will be one of revolution. If $\rho = \varphi(t)$ be the vector equation of the curve, referred to a point in the axis of revolution, $\mathrm{T}\rho'\mathrm{d}t$ will be the element of arc of the meridian (or generating curve). Let u be the distance of a point of the curve from the axis of revolution, and φ the angle of revolution. Then $u\mathrm{d}\varphi$ will be the element of the parallel of latitude, and $u\mathrm{d}\varphi\mathrm{T}\rho'\mathrm{d}t$ the element of area. Hence, if α is a unit-vector parallel to the axis, the area of a portion of a zone will be

$$S = \Phi \int_{t_0}^{t} u\mathrm{T}\varrho' = \Phi \int_{t_0}^{t} \mathrm{T}.\varrho'\mathrm{V}\alpha\varrho. \tag{5}$$

4°. The direction of the normal to a surface of revolution is the same as that of the perpendicular from the origin (which we still suppose to be in the axis) dropped on the tangent to the meridian. If ϱ' be the tangent to the meridian, the projection of the radius vector ϱ on that tangent is

$$\frac{1}{\rho'}\ \mathrm{S}\varrho\varrho', \text{ and its length } \frac{\mathrm{TS}\rho\rho'}{\mathrm{T}\rho'}.$$

Represent the perpendicular from the origin on the tangent by $\boldsymbol{\nu}$. Then if $\mathrm{T}\nu$ be the length intercepted by the tangent,

$$\begin{aligned} \mathrm{T}\nu^2 &= \mathrm{T}^2\varrho - \frac{\mathrm{S}^2\rho\rho'}{\mathrm{T}^2\rho'}, \\ &= \frac{\mathrm{T}^2\rho\rho' - \mathrm{S}^2\rho\rho'}{\mathrm{T}^2\rho'} = \frac{-\mathrm{V}^2\rho\rho'}{\mathrm{T}^2\rho'} = \frac{\mathrm{T}^2\mathrm{V}\rho\rho'}{\mathrm{T}^2\rho'}, \end{aligned}$$

and

$$\mathrm{T}\nu = \frac{\mathrm{TV}\rho\rho'}{\mathrm{T}\rho'}. \tag{6}$$

This expression gives the length of the perpendicular from the origin on the tangent plane, and hence the element of volume swept by the radius vector is given by

$$\tfrac{1}{3}\ \mathrm{T}\nu.u\mathrm{d}\varphi\mathrm{T}\varrho'\mathrm{d}t = \tfrac{1}{3}\ \mathrm{TV}\varrho\varrho'.u\mathrm{d}\varphi\mathrm{d}t.$$

The finite volume will therefore be expressed by

$$V = \tfrac{1}{3}\ \Phi \int_{t_0}^{t} u.\mathrm{TV}\varrho\varrho'. \tag{7}$$

5°. Write $\varrho = \varphi(t, u)$ as the equation of a surface, with the condition that t and u are two scalar indeterminates, the changes in t determining a series of successive curves on the surface, these curves intersected by another series of successive curves determined by the changes in u. Then

$$\mathrm{D}_t\varrho = \varrho'_1$$

will be the tangent to a curve of the first series, and

$$\mathrm{D}_u\varrho = \varrho'_2$$

will be the tangent to a curve of the second series. The element of area of the surface will therefore be $\mathrm{TV}\varrho'_1\varrho'_2\mathrm{d}u\mathrm{d}t$, and the finite area is

$$S = \int_u \int_t \mathrm{TV}\varrho'_1\varrho'_2. \tag{8}$$

6°. The direction of the normal to the surface is given by $V\varrho'_1\varrho'_2$, and the projection of the radius vector on the normal is

$$\nu = \frac{1}{V\rho'_1\rho'_2} S\varrho V\varrho'_1\varrho'_2 = \frac{1}{V\rho'_1\rho'_2} S\varrho\varrho'_1\varrho'_2, \tag{9}$$

and this is the perpendicular from the origin on the tangent plane to the surface at the extremity of ϱ. Its length is

$$T\nu = \pm \frac{S\rho\rho'_1\rho'_2}{TV\rho'_1\rho'_2}. \tag{10}$$

Hence, the element of volume swept by the radius vector is

$$\tfrac{1}{3}\, TV\varrho'_1\varrho'_2 \cdot \frac{S\rho\rho'_1\rho'_2}{TV\rho'_1\rho'_2}\, du dt,$$

and the finite volume is

$$V = \tfrac{1}{3} \int_u \int_t S\varrho\varrho'_1\varrho'_2. \tag{11}$$

Here, again, as in 2°, a change of origin will affect the result, and give a different portion of the volume. As before, suppose δ to be the vector of the new origin. Then the new vector, whose extremity generates the surface, is $\varpi = \varrho - \delta$, and the volume swept by ϖ is

$$V = \tfrac{1}{3} \int_u \int_t S\varpi\varpi'_1\varpi'_2 = \tfrac{1}{3} \int_u \int_t S(\varrho - \delta)\varrho'_1\varrho'_2. \tag{12}$$

7°. The equation of the ellipse may be written

$$\varrho = \alpha \cos x + \beta \sin x, \tag{13}$$

where α and β are the principal semi-diameters, and x is the eccentric angle. $T\alpha = a$, $T\beta = b$, and α is perpendicular to β. By differentiation

$$D_x\varrho = \varrho' = -\alpha \sin x + \beta \cos x,$$

and taking the tensor

$$T\varrho' = \sqrt{a^2 \sin^2 x + b^2 \cos^2 x}$$
$$= \sqrt{(a^2 - b^2) \sin^2 x + b^2}.$$

Hence the arc of the ellipse is

$$s - s_0 = \int_{x_0}^{x} T\varrho' = \int_{x_0}^{x} \sqrt{(a^2 - b^2) \sin^2 x + b^2}. \tag{14}$$

The integration of this function, of course, involves elliptic functions For the circle $a = b$, and

$$s - s_0 = \int_{x_0}^{x} b = b(x - x_0).$$

8°. The equation of the hyperbola may be written, x being a variable analogous to the eccentric angle,

$$\varrho = \alpha \operatorname{Ch} x + \beta \operatorname{Sh} x. \qquad (15)$$

By differentiation

$$\varrho' = \alpha \operatorname{Sh} x + \beta \operatorname{Ch} x,$$

$$\mathrm{T}\varrho' = \sqrt{a^2 \operatorname{Sh}^2 x + b^2 \operatorname{Ch}^2 x}$$

$$= \sqrt{(a^2 + b^2) \operatorname{Sh}^2 x + b^2}.$$

Hence the expression for the length of the arc is

$$s - s_0 = \int_{x_0}^{x} \sqrt{(a^2 + b^2) \operatorname{Sh}^2 x + b^2}. \qquad (16)$$

In the equilateral hyperbola $a = b$, and

$$s - s_0 = \int_{x_0}^{x} a \sqrt{2 \operatorname{Sh}^2 x + 1} = a \int_{x_0}^{x} \sqrt{\operatorname{Ch} 2x}.$$

The equation of the hyperbola referred to its asymptotes is

$$\varrho = x\alpha + x^{-1}\beta. \qquad (17)$$

Whence by differentiation

$$\varrho' = \alpha - x^{-2}\beta.$$

Let $\mathrm{T}\alpha = k = \mathrm{T}\beta$. Then

$$\mathrm{T}\varrho' = k \sqrt{1 - 2x^{-2}\mathrm{SU}\alpha\beta + x^{-4}},$$

and

$$s - s_0 = k \int_{x_0}^{x} \frac{\sqrt{x^4 + 2x^2 \cos \frac{\beta}{\alpha} + 1}}{x^2}. \qquad (18)$$

In the equilateral hyperbola, $\cos \frac{\beta}{\alpha} = 0$, and

$$s - s_0 = k \int_{x_0}^{x} \frac{\sqrt{x^4 + 1}}{x^2}.$$

9°. The equation of the parabola may be written

$$\rho = \frac{x^2}{2}\,\alpha + x\beta, \qquad (19)$$

where $\alpha \perp \beta$. Then

$$\rho' = x\alpha + \beta,$$

and if $T\alpha = a$, and $T\beta = b$,

$$T\rho' = \sqrt{a^2x^2 + b^2},$$

and

$$s - s_0 = \int_{x_0}^{x} \sqrt{a^2x^2 + b^2}$$

$$= \left[\frac{x}{2}\sqrt{a^2x^2 + b^2} + \frac{b^2}{2\,a}\,\mathrm{Sh}^{-1}\,\frac{ax}{b}\right]_{x_0}^{x}$$

$$= \frac{b^2}{2a}\left[\frac{ax}{b}\sqrt{\frac{a^2x^2}{b^2} + 1} + \mathrm{Sh}^{-1}\,\frac{ax}{b}\right]_{x_0}^{x}.$$

Let $v = \mathrm{Sh}^{-1}\,\frac{ax}{b}$; then the expression for the arc becomes

$$s - s_0 = \frac{b^2}{4a}\left[\mathrm{Sh}\;2v + 2v\right]_{v_0}^{v}. \qquad (20)$$

10°. The equation of the helix on the elliptic cylinder (or, rather, of a curve analogous to the helix) is

$$\rho = \alpha \cos x + \beta \sin x + \gamma x, \qquad (21)$$

where $\alpha \perp \beta \perp \gamma \perp \alpha$, α and β being semi-diameters of the elliptic base. By differentiation,

$$\rho' = -\,\alpha \sin x + \beta \cos x + \gamma;$$

$$T\rho' = \sqrt{(a^2 \sin^2 x + b^2 \cos^2 x + c^2)},$$

if $T\alpha = a$, $T\beta = b$, $T\gamma = c$.

$$\therefore\; s - s_0 = \int_{x_0}^{x} \sqrt{(a^2 - b^2)\sin^2 x + b^2 + c^2}. \qquad (22)$$

This integration, as it should, involves elliptic functions. If the base of the cylinder be circular, then $a = b$, and

$$s - s_0 = \int_{x_0}^{x} \sqrt{b^2 + c^2} = \sqrt{b^2 + c^2}\,(x - x_0) \qquad (23)$$

11°. The scalar equations of the cycloid are

$$x = a\,(\theta - \sin\theta),$$

$$y = a\,(1 - \cos\theta).$$

Let $T\alpha = T\beta = a$. Then the vector equation of the cycloid may be written

$$\rho = xU\alpha + yU\beta = \alpha(\theta - \sin\theta) + \beta(1 - \cos\theta). \qquad (24)$$

Whence

$$\rho' = \alpha(1 - \cos\theta) + \beta\sin\theta,$$

$$\rho'^2 = \alpha^2(1 - \cos\theta)^2 + \beta^2\sin^2\theta,$$

$$T\rho' = a\sqrt{1 - 2\cos\theta + \sin^2\theta + \cos^2\theta}$$

$$= a\sqrt{2 - 2\cos\theta} = 2a\sin\tfrac{1}{2}\theta.$$

$$\therefore\quad s - s_0 = 2a\int_{\theta_0}^{\theta}\sin\tfrac{1}{2}\theta = 4a\Big[\cos\tfrac{1}{2}\theta\Big]_{\theta}^{\theta_0}. \qquad (25)$$

For the complete arc

$$s - s_0 = 4a\Big[\cos\tfrac{1}{2}\theta\Big]_{2\circlearrowleft}^{0} = 8a.$$

12°. The areas of the conic sections are easily obtained from formulas (3) and (4). For the ellipse we have, α and β being any two conjugate semi-diameters,

$$\rho = \alpha\cos x + \beta\sin x,$$

$$\rho' = -\alpha\sin x + \beta\cos x,$$

whence

$$TV\rho\rho' = TV(\alpha\beta\cos^2 x - \beta\alpha\sin^2 x) = TV\alpha\beta;$$

$$\therefore\quad A = \tfrac{1}{2}\int_{x_0}^{x} TV\rho\rho' = \frac{TV\alpha\beta}{2}\Big[x - x_0\Big]$$

$$= \frac{ab}{2}\sin\frac{\beta}{\alpha}\Big[x - x_0\Big]. \qquad (26)$$

Or, if $\alpha \perp \beta$,

$$A = \frac{ab}{2}\Big[x - x_0\Big].$$

The whole area is $\circlearrowleft ab$.

We may change the origin to any fixed point, and so obtain the area of any portion of the ellipse. Suppose ε is the vector of the fixed point, and ϖ the new radius vector from this point; then the equation of the ellipse becomes

$$\varpi = (\rho - \varepsilon) = \alpha\cos x + \beta\sin x - \varepsilon.$$

Whence

$$\varpi' = -\alpha \sin x + \beta \cos x = \varrho',$$

$$\mathrm{TV}\varpi\varpi' = \mathrm{TV}\ (\alpha\beta + \varepsilon\alpha \sin x - \varepsilon\beta \cos x),$$

and

$$A = \tfrac{1}{2}\int_{x_0}^{x} \mathrm{TV}(\alpha\beta + \varepsilon\alpha \sin x - \varepsilon\beta \cos x). \qquad (27)$$

If the new origin lie anywhere on α or β, one of the terms of this integration disappears. Suppose the new origin is on α; then $\varepsilon = m\alpha$, where m is a scalar. Hence

$$A = \tfrac{1}{2}\int_{x_0}^{x} \mathrm{TV}\alpha\beta(1 - m \cos x)$$

$$= \frac{ab}{2} \sin \frac{\beta}{\alpha} \Big[x - m \sin x\Big]_{x_0}^{x}. \qquad (28)$$

13°. The equation of the hyperbola,

$$\varrho = \alpha \mathrm{Ch}x + \beta \mathrm{Sh}x,$$

will give results precisely similar to those of the preceding section, with the hyperbolic sine and cosine everywhere substituted for the circular. With the origin of vectors at the centre, the area swept by the radius vector is

$$A = \frac{ab}{2} \sin \frac{\beta}{\alpha} \Big[x - x_0\Big]; \qquad (29)$$

or if $\alpha \perp \beta$

$$A = \frac{ab}{2} \Big[x - x_0\Big].$$

This is the area of a portion of the surface exterior to the curve, contained between the curve and the limiting positions of the radius vector. To find a portion of the inner area we need only transfer the origin to some point on or within the curve, and proceed as before. Suppose the origin to lie on α, at a distance $m\mathrm{T}\alpha$ from the centre. The finite area is

$$A = \frac{ab}{2} \sin \frac{\beta}{\alpha} \Big[x - m\mathrm{Sh}x\Big]_{x_0}^{x}. \qquad (30)$$

In formulae (28) and (30), put $m = \cos x_m$ and $m = \mathrm{Ch}x_m$, and take x_m and 0 for the limits. Then if $\alpha \perp \beta$, the elliptic area (doubled) is

$$A_1 = ab[x_m - \cos x_m \sin x_m], \qquad (31)$$

and the hyperbolic area (doubled) is

$$A_2 = ab[x_m - \mathrm{Ch}x_m\mathrm{Sh}x_m]. \tag{32}$$

These areas are those of segments cut by lines parallel to β.

14°. In the equation of the parabola

$$\varrho = \frac{x^2}{2}\alpha + x\beta,$$

let $\mathrm{T}\alpha = a$, $\mathrm{T}\beta = b$. In this equation of the curve, α is the diameter and β is the tangent at the origin. We have

$$\varrho' = x\alpha + \beta,$$

and

$$\mathrm{TV}\varrho\varrho' = \mathrm{TV}(\frac{x^2}{2}\alpha\beta - x^2\alpha\beta) = \mathrm{TV}\alpha\beta\,\frac{x^2}{2},$$

$$A = \frac{ab}{4}\sin{}_{\alpha}^{\beta}\int_{x_0}^{x} x^2 = \frac{ab\sin{}_{\alpha}^{\beta}}{12}[x^3 - x_0{}^3]; \tag{33}$$

this being the area of a sector of which the vertex is at the origin. If $\alpha \perp \beta$, then

$$A = \frac{ab}{12}\left[x^3 - x_0{}^3\right].$$

Transfer the origin to any point on the tangent β. Let

$$\varepsilon = m\beta,$$

where m is a scalar. The new equation becomes

$$(\varrho - \varepsilon) = \varpi = \frac{x^2}{2}\alpha + (x - m)\beta.$$

Whence

$$\varpi' = x\alpha + \beta,$$

$$\mathrm{TV}\varpi\varpi' = \mathrm{TV}\left[\frac{x^2}{2}\alpha\beta - (x^2 - xm)\alpha\beta\right]$$

$$= \mathrm{TV}\alpha\beta\left(xm - \frac{x^2}{2}\right),$$

and

$$A = \frac{ab\sin{}_{\alpha}^{\beta}}{2}\int_{x_0}^{x}(xm - \frac{x^2}{2})$$

$$= \frac{ab\sin{}_{\alpha}^{\beta}}{2}\left[\frac{x^2m}{2} - \frac{x^3}{6}\right]_{x_0}^{x}. \tag{34}$$

If the sector be cut with one boundary parallel to α, so that at the upper limit $x = m$, while x_0 is taken $= 0$, then

$$A = \frac{ab}{2} \sin{}^{\beta}_{\alpha} \left[\frac{x^3}{2} - \frac{x^3}{6}\right] = \frac{abx^3}{6} \sin{}^{\beta}_{\alpha}. \qquad (35)$$

This is one-third of the parallelogram formed by the co-ordinates of the point at the extremity of ϱ.

15°. The equation of the cycloid gives the following:—

$$\varrho = \alpha(\theta - \sin\theta) + \beta(1 - \cos\theta),$$

$$\varrho' = \alpha(1 - \cos\theta) + \beta\sin\theta,$$

$$\mathrm{TV}\varrho\varrho' = \mathrm{TV}[\alpha\beta(\theta\sin\theta - \sin^2\theta - \{1 - \cos\theta\}^2)]$$

$$= a^2(\theta\sin\theta - 4\sin^2\tfrac{1}{2}\theta).$$

Hence

$$A = \frac{a^2}{2}\int_{\theta_0}^{\theta}(\theta\sin\theta + 2\cos\theta - 2)$$

$$= \frac{a^2}{2}\Big[\sin\theta - \theta\cos\theta + 2\sin\theta - 2\theta\Big]_{\theta_0}^{\theta}$$

$$= \frac{a^2}{2}\Big[3\sin\theta - \theta\cos\theta - 2\theta\Big]_{\theta_0}^{\theta}. \qquad (36)$$

The complete area is $A = 3\odot a^2$.

16°. In 14°, we found for the parabola

$$\varrho = \frac{x^2}{2}\alpha + x\beta,$$

$$\mathrm{T}\varrho' = \sqrt{a^2x^2 + b^2}.$$

Revolve the curve about β, the tangent at the vertex. Make

$$u = \frac{ax^2}{2}.$$

The area of the surface thus generated will be [see (5)]

$$S = \Phi\int_{x_0}^{x} u\mathrm{T}\varrho' = \Phi\frac{a}{2}\int_{x_0}^{x} x^2\sqrt{a^2x^2 + b^2}$$

$$= \Phi\left[\frac{(2a^2x^2 + b^2)x\sqrt{a^2x^2 + b^2}}{16a} - \frac{b^4}{16a^2}\,\mathrm{Sh}^{-1}\frac{ax}{b}\right]_{x_0}^{x}$$

$$= \Phi \frac{b^4}{64a^2}\Big[\mathrm{Sh}4v - 4v\Big]_{v_0}^{v}, \qquad (37)$$

where $v = \mathrm{Sh}^{-1}\frac{ax}{b}$.

If we revolve the curve about the diameter α, we may then put $u = bx$, and the area of the surface generated will be found to be

$$S = \Phi b \int_{x_0}^{x} x\sqrt{a^2x^2 + b^2} = \Phi b \left[\frac{(a^2x^2 + b^2)^{\frac{3}{2}}}{3\,a^2}\right]_{x_0}^{x}$$

$$= \Phi \frac{b^4}{3a^2}\Big[\mathrm{Ch}^3 v\Big]_{v_0}^{v}, \qquad (38)$$

where $v = \mathrm{Sh}^{-1}\frac{ax}{b}$.

17°. Again, in 14°, we found for the parabola, when $\alpha \perp \beta$,

$$\mathrm{TV}\varrho\varrho' = \frac{ab}{2}\,x^2.$$

Whence, for the volume of the solid of revolution (generated by the revolution of a sector), the axis being the tangent at the vertex, we have, making $u = \frac{a}{2}\,x^2$,

$$V = \tfrac{1}{3}\,\Phi \int_{x_0}^{x} u.\mathrm{TV}\varrho\varrho' = \Phi\,\frac{a^2b}{12}\int_{x_0}^{x} x^4 = \Phi\,\frac{a^2b}{60}\Big[x^5 - x_0^{\,5}\Big]; \quad (39)$$

and for the volume of the paraboloid whose axis is the diameter, making $u = bx$,

$$V = \Phi\,\frac{ab^2}{6}\int_{x_0}^{x} x^3 = \Phi\,\frac{ab^2}{24}\Big[x^4 - x_0^{\,4}\Big]. \qquad (40)$$

18°. For the cycloid we have found, in 11°,

$$\varrho = \alpha(\theta - \sin\theta) + \beta(1 - \cos\theta),$$

$$\mathrm{T}\varrho' = 2a\,\sin\tfrac{1}{2}\theta.$$

If we revolve it about the base, we shall have $u = a(1 - \cos\theta)$, and the area of the surface generated will be

$$S = \Phi \int_{\theta_0}^{\theta} u\mathrm{T}\varrho' = 2\,\Phi a^2 \int_{\theta_0}^{\theta} \sin\tfrac{1}{2}\,\theta\,(1 - \cos\theta)$$

$$= 8\,\Phi a^2 \int_{\frac{1}{2}\theta_0}^{\frac{1}{2}\theta} \sin^3 \tfrac{1}{2}\,\theta = 8\Phi a^2 \Big[\tfrac{1}{3}\cos^3 \tfrac{1}{2}\,\theta - \cos \tfrac{1}{2}\,\theta\Big]_{\theta_0}^{\theta} . \quad (41)$$

The complete area is $S = \frac{64}{3}\,\eth a^2$.

The area of the surface formed by revolving the cycloid about β would be found by putting $u = a(\theta - \sin\theta)$ and using formula (5) as before.

19°. The volume generated by revolving a sector of the cycloid about its base is

$$V = \frac{\Phi}{3}\int_{\theta_0}^{\theta} u.\mathrm{TV}\varrho\varrho' = \Phi\,\frac{a^3}{3}\int_{\theta_0}^{\theta}(1-\cos\theta)(\theta\sin\theta + 2\cos\theta - 2)$$

$$= \Phi\,\frac{a^3}{3}\int_{\theta_0}^{\theta}(\theta\sin\theta + 4\cos\theta - \tfrac{1}{2}\,\theta\sin 2\,\theta - 2\cos^2\theta - 2)$$

$$= \Phi\,\frac{a^3}{3}\Big[5\sin\theta - \frac{5}{8}\sin 2\,\theta - \theta(\cos\theta - \tfrac{1}{4}\cos 2\,\theta + 3)\Big]_{\theta_0}^{\theta} . \quad (42)$$

The complete volume is $5\,\eth^2 a^3$.

20°. In 7°, we have found for the ellipse

$$\mathrm{T}\varrho' = \sqrt{(a^2 - b^2)\sin^2 x + b^2} = \sqrt{a^2 - (a^2 - b^2)\cos^2 x}.$$

The prolate ellipsoid is generated by revolving the ellipse about its major axis. In this case, the equation being $\varrho = \alpha\cos x + \beta\sin x$, we shall have

$$u = b\sin x,$$

and for the surface

$$S = \Phi\int_{x_0}^{x} u.\mathrm{T}\varrho' = \Phi b\int_{x_0}^{x}\sin x\sqrt{a^2 - (a^2 - b^2)\cos^2 x}$$

$$= \Phi b\int_{\cos x_0}^{\cos x}\sqrt{a^2 - c^2\cos^2 x},$$

where $c^2 = a^2 - b^2$. Whence

$$S = \Phi b\Big[\frac{\cos x}{2}\sqrt{a^2 - c^2\cos^2 x} + \frac{a^2}{2c}\sin^{-1}\frac{c\cos x}{a}\Big]_{x_0}^{x}$$

$$= \Phi \frac{a^2 b}{2c}\left[\frac{c \cos x}{a}\sqrt{1 - \frac{c^2 \cos^2 x}{a^2}} + \sin^{-1}\frac{c \cos x}{a}\right]_{x_0}^{x}.$$

Let $v = \sin^{-1}\frac{c \cos x}{a}$; then

$$S = \Phi \frac{a^2 b}{4c}\Big[\sin 2v + 2v\Big]_{v_0}^{v}. \qquad (43)$$

For the oblate ellipsoid of revolution, the substitution

$$u = a \cos x$$

gives, by the same method of proceeding as before,

$$S = \Phi a \int_{x_0}^{v} \cos x \sqrt{c^2 \sin^2 x + b^2} = \Phi a \int_{\sin x_0}^{\sin x} \sqrt{c^2 \sin^2 x + b^2}$$

$$= \Phi a \left[\frac{\sin x}{2}\sqrt{c^2 \sin^2 x + b^2} + \frac{b^2}{2c}\,\mathrm{Sh}^{-1}\frac{c \sin x}{b}\right]_{x_0}^{x}$$

$$= \Phi \frac{ab^2}{2c}\left[\frac{c \sin x}{b}\sqrt{\frac{c^2 \sin^2 x}{b^2} + 1} + \mathrm{Sh}^{-1}\frac{c \sin x}{b}\right]_{x_0}^{x}.$$

Let $v = \mathrm{Sh}^{-1}\frac{c \sin x}{b}$; then

$$S = \Phi \frac{ab^2}{4c}\Big[\mathrm{Sh}\, 2v + 2v\Big]_{v_0}^{v}. \qquad (44)$$

21°. From the equation of the hyperbola we found (8°),

$$\mathrm{T}\varrho' = \sqrt{c^2\mathrm{Sh}^2 x + b^2} = \sqrt{c^2\mathrm{Ch}^2 x - a^2},$$

where $c^2 = a^2 + b^2$. Hence, for the unparted hyperboloid of revolution, generated by revolving the hyperbola about the axis β, we put

$$u = a\mathrm{Ch}x,$$

and find

$$S = \Phi \int_{x_0}^{x} u.\mathrm{T}\varrho' = \Phi a \int_{x_0}^{x} \mathrm{Ch}x\sqrt{c^2\mathrm{Sh}^2 x + b^2}$$

$$= \Phi \frac{ab^2}{2c}\left[\frac{c\mathrm{Sh}x}{b}\sqrt{\frac{c^2\mathrm{Sh}^2 x}{b^2} + 1} + \mathrm{Sh}^{-1}\frac{c\mathrm{Sh}x}{b}\right]_{x_0}^{x}$$

$$= \Phi \frac{ab^2}{4c}\Big[\mathrm{Sh}2v + 2v\Big]_{v_0}^{v}, \qquad (45)$$

where $v = \mathrm{Sh}^{-1}\frac{c\mathrm{Sh}x}{b}$.

Again, the substitution $u = b\mathrm{Sh}x$ gives, for the area of the parted hyperboloid of revolution,

$$S = \Phi b \int_{x_0}^{x} \mathrm{Sh}x\sqrt{c^2\mathrm{Ch}^2x - a^2}$$

$$= \Phi\,\frac{a^2b}{2c}\left[\frac{c\mathrm{Ch}x}{a}\sqrt{\frac{c^2\mathrm{Ch}^2x}{a^2} - 1} - \mathrm{Ch}^{-1}\,\frac{c\mathrm{Ch}x}{a}\right]_{x_0}^{x}$$

$$= \Phi\,\frac{a^2b}{4c}\Big[\mathrm{Sh}\;2v - 2v\Big]_{v_0}^{v}, \tag{46}$$

where $v = \mathrm{Ch}^{-1}\,\dfrac{c\mathrm{Ch}x}{a}$.

22°. By the application of formula (7) the volume generated by the revolution of a sector of a conic about a principal axis may be obtained very simply. With $\alpha \perp \beta$, the equations of the ellipse and hyperbola give

$$\mathrm{TV}\rho\rho' = \mathrm{TV}\alpha\beta = ab.$$

The substitution $u = b \sin x$ gives

$$V = \tfrac{1}{3}\,\Phi \int u\mathrm{TV}\rho\rho' = \tfrac{1}{3}\,\Phi\; ab^2 \int_{x_0}^{x} \sin x$$

$$= \tfrac{1}{3}\,\Phi\; ab^2\,\Big[\cos\,x\Big]_{x}^{x_0}, \tag{47}$$

for the volume enclosed by the prolate ellipsoid; and the substitution $u = a \cos x$, gives

$$V = \tfrac{1}{3}\,\Phi\; a^2b \int_{x_0}^{x} \cos\,x = \tfrac{1}{3}\,\Phi\; a^2b\Big[\sin\,x\Big]_{x_0}^{x}, \tag{47'}$$

for the volume enclosed by the oblate ellipsoid.

23°. Similarly, the substitution $u = a\mathrm{Ch}x$ gives

$$V = \Phi\,\frac{a^2b}{3}\int_{x_0}^{x}\mathrm{Ch}x = \Phi\,\frac{a^2b}{3}\Big[\mathrm{Sh}x\Big]_{x_0}^{x}, \tag{48}$$

for the volume swept by a central sector in generating the unparted hyperboloid; and the substitution $u = b\mathrm{Sh}x$ gives

$$V = \Phi\,\frac{ab^2}{3}\int_{x_0}^{x}\mathrm{Sh}x = \Phi\,\frac{ab^2}{3}\Big[\mathrm{Ch}x\Big]_{x_0}^{x}, \tag{48'}$$

for the volume swept by a central sector in generating the parted hyperboloid.

The volumes enclosed by the surfaces of the hyperboloids will be considered later in connection with the discussion of the general equations of the quadric surfaces.

To obtain the volumes bounded by the general ellipsoids and elliptic hyperboloids, it is only necessary, in the results of 22° and 23°, by a well-known principle, to change a^2 into ac, or b^2 into bc, where c is the length of the third semi-diameter.

24°. The general discussion of the areas and volumes of a very important class of surfaces will be facilitated by writing, in each case, a quaternion equation of the surface in terms of two independent scalar variables.

Let α, β, γ — three vectors diverging from the origin — be any three axes of the surface whose generating vector is ϱ. Let $\sigma =$ the vector — complanar with ϱ and α — whose extremity describes the section of the surface in the plane of $\beta\gamma$. Write the equation of this section in the form

$$\sigma = \beta f_3 y + \gamma f_4 y,$$

and the equation of the surface in the form

$$\varrho = \alpha f_1 x + \sigma f_2 x, \qquad (49)$$

where f_1, f_2, etc., are separately functions of a single variable. In this equation (49), the two scalar variables are evidently independent of each other, a new value of either determining a new point on the surface; ϱ will describe, if x remain constant, a section parallel to the plane of $\beta\gamma$, if y remain constant, a section whose plane contains α. For convenience we may suppose $\alpha \perp \beta \perp \gamma$. We shall now have

$D_x\varrho = \varrho'_1 =$ tangent to meridian at extremity of ϱ,

$D_y\varrho = \varrho'_2 =$ tangent to parallel of latitude at extremity of ϱ; and, performing the differentiation,

$$\varrho'_1 = \alpha f_1' x + \sigma f_2' x, \quad \varrho'_2 = \sigma' f_2 x,$$

whence

$$V\varrho'_1\varrho'_2 = V\alpha\sigma' f_2 x f_1' x + V\sigma\sigma' f_2 x f_2' x,$$

and, since $V\alpha\sigma'$ and $V\sigma\sigma'$ are perpendicular to each other,

$$V^2\varrho'_1\varrho'_2 = (f_2 x)^2[V^2\alpha\sigma'(f_1' x)^2 + V^2\sigma\sigma'(f_2' x)^2],$$

or

$$TV\varrho'_1\varrho'_2 = f_2 x \sqrt{[T^2\alpha\sigma'(f_1' x)^2 + T^2V\sigma\sigma'(f_2' x)^2]}. \qquad (50)$$

This result, substituted in formula (8), gives

$$S = \int_y \int_x f_2 x \sqrt{[\mathrm{T}^2 \alpha\sigma' (f_1' x)^2 + \mathrm{T}^2 \mathrm{V}\sigma\sigma' (f_2' x)^2]}. \qquad (51)$$

Let us now write a system of equations for the quadric surfaces. For this purpose we have only to substitute, in equation (49), such functions of x and of y as shall cause σ, when y varies, and ϱ, when y is constant and x varies, to describe conic sections; and any functions which identically satisfy the conditions $(f_1 x)^2 \pm (f_2 x)^2 = 1$, $(f_3 x)^2 \pm (f_4 x)^2 = 1$, or any equation of the second degree in $f_1 x$, $(f_2 x . f_3 y)$, and $(f_2 x . f_4 y)$, will evidently serve our purpose. We may write then, for a system of quadric surfaces, the following equations:—

I. $\varrho = \alpha \cos x + \sigma \sin x$,
$\sigma = \beta \cos y + \gamma \sin y$; the ellipsoid;

II. $\varrho = \alpha \mathrm{Ch} x + \sigma \mathrm{Sh} x$,
$\sigma = \beta \cos y + \gamma \sin y$; the parted hyperboloid, having α for its principal axis;

III. $\varrho = \alpha \mathrm{Sh} x + \sigma \mathrm{Ch} x$,
$\sigma = \beta \cos y + \gamma \sin y$; the unparted hyperboloid, having α for its principal axis;

IV. $\varrho = \alpha \frac{x^2}{2} + \sigma x$,
$\sigma = \beta \cos y + \gamma \sin y$; the elliptic paraboloid, having α for its axis;

V. $\varrho = \alpha \frac{x^2}{2} + \sigma x$,
$\sigma = \beta \mathrm{Ch} y + \gamma \mathrm{Sh} y$; the hyperbolic paraboloid, having α for its axis;

and finally I write

VI. $\varrho = \alpha x + \sigma \frac{x^2}{2}$,
$\sigma = \beta \cos y + \gamma \sin y$, as being analogous to IV. and V., though not representing a quadric.

The function $\mathrm{TV}\varrho'_1 \varrho'_2$ derived from I.,—if we observe that $\mathrm{V}\sigma\sigma' = \beta\gamma$, and make the substitutions $f_1 x = \cos x$, $f_1' x = -\sin x$, etc., in (50),—now becomes

$$\mathrm{TV}\varrho'_1 \varrho'_2 = \sin x \sqrt{(a^2 s^2 \sin^2 x + b^2 c^2 \cos^2 x)}$$

$$= s \sin x \sqrt{\left(a^2 - \frac{a^2 s^2 - b^2 c^2}{s^2} \cos^2 x\right)},$$

where $a = T\alpha$, $b = T\beta$, $c = T\gamma$, $s = T\sigma'$. Hence, for the general ellipsoid,

$$S = \int_y s \int_x \sin x \sqrt{(a^2 - \frac{a^2s^2 - b^2c^2}{s^2} \cos^2 x)}$$

$$= - \int_y s . \int_{\cos x} \sqrt{(a^2 - \frac{a^2s^2 - b^2c^2}{s^2} \cos^2 x)},$$

and, if $m^2 = \frac{a^2s^2 - b^2c^2}{a^2s^2}$,

$$S = \int_y \frac{sa}{4m} \Big[2m \cos x \sqrt{(1 - m^2 \cos x)} + 2 \sin^{-1}(m \cos x) \Big]_{x_0}^{x},$$

I′. $$\text{or } S = \int_y \frac{sa}{4m} \Big[\sin 2v + 2v \Big]_{v_0}^{v},$$

where $v = \sin^{-1} (m \cos x)$.

If the surface be a prolate ellipsoid of revolution, then $b = c = s$. $m^2 = \frac{a^2 - b^2}{a^2}$, and

$$S = \Big[y - y_0 \Big] \frac{ab}{4m} \Big[\sin 2v + 2v \Big]_{v_0}^{v},$$

a result which agrees with that obtained in 20°.

Repeating this process of integration for cases II. and III., we obtain in succession

II′. $$S = \int_y \frac{sa}{4m} \Big[\text{Sh} 2v - 2v \Big]_{v_0}^{v},$$

where $v = \text{Ch}^{-1} (m \text{Ch} x)$ and $m^2 = \frac{a^2s^2 + b^2c^2}{a^2s^2}$;

III′. $$S = \int_y \frac{sa}{4m} \Big[\text{Sh} 2v + 2v \Big]_{v_0}^{v},$$

where $v = \text{Sh}^{-1} (m \text{Sh} x)$ and $m^2 = \frac{b^2c^2 - a^2s^2}{a^2s^2}$.

For cases IV. and V., we shall have $\varrho'_1 = \alpha x + \sigma$, $\varrho'_2 = \sigma' x$, and

$$V \varrho'_1 \varrho'_2 = \alpha \sigma' x^2 + \beta \gamma x,$$

whence

IV′. $$S = \int_y \int_x x \sqrt{(a^2s^2x^2 + b^2c^2)} = \int_y \Big[\frac{(a^2s^2x^2 + b^2c^2)^{\frac{3}{2}}}{3a^2s^2} \Big]_{x_0}^{x}$$

$$= \int_y \frac{b^3c^3}{3a^2s^2} \Big[\mathrm{Ch}^3 v \Big]_{v_0}^{v},$$

where $v = \mathrm{Sh}^{-1} \frac{asx}{bc}$;

V′. = IV′. with a different value for s; and finally VI. gives $\varrho'_1 = \sigma x + \alpha$, $\varrho'_2 = \frac{1}{2} \sigma' x^2$, whence

$$V \varrho'_1 \varrho'_2 = \beta\gamma \frac{x^3}{2} + \alpha\sigma' \frac{x^2}{2},$$

and

$$\text{VI}'. \quad S = \int_y \int_x \frac{x^2}{2} \sqrt{(b^2c^2x^2 + a^2s^2)} = \int_y \frac{a^4s^4}{64b^3c^3} \Big[\mathrm{Sh} 4v - 4v \Big]_{v_0}^{v},$$

where $v = \mathrm{Sh}^{-1} \frac{bcx}{as}$. The equations for the surface generated by revolving the parabola about its tangent at the vertex, will be found from VI. and VI′. by putting $b = c = s$, and its finite area is

$$S = \Big[y - y_0 \Big] \frac{a^4}{64 . b^2} \Big[\mathrm{Sh} 4v - 4v \Big]_{v_0}^{v},$$

where $v = \mathrm{Sh}^{-1} \frac{bx}{a}$.

25°. Various forms of equations may be used for the quadrics. Thus, the parted hyperboloid represented by III. may also be represented by the equations

$$\varrho = \beta \mathrm{Sh} x + \sigma \mathrm{Ch} x,$$

$$\sigma = \alpha \mathrm{Ch} y + \gamma \mathrm{Sh} y.$$

Again, the unparted hyperboloid represented by IV. can also be repsented by

$$\varrho = \beta \cos x + \sigma \sin x,$$

$$\sigma = \alpha \mathrm{Sh} y + \gamma \mathrm{Ch} y,$$

or by

$$\varrho = \pm \beta \mathrm{Ch} x + \sigma \mathrm{Sh} x,$$

$$\sigma = \alpha \mathrm{Ch} y + \gamma \mathrm{Sh} y.$$

But, with regard to the last two sets of equations, it is to be observed that they must be taken together, in order to give a complete representation of the surface, for real values of the variables x and y. For, if we use only real values of the variables, the first set of equations gives no points of the surface exterior to a pair of planes parallel to the plane of $\alpha\gamma$, and passing through the extremities of $+\beta$ and $-\beta$; whereas the second set gives no points of the surface contained between these planes. These might be called, therefore, *supplemental equations* of the surface.

If the equations be reduced to scalar forms in rectangular co-ordinates, they become identical, and their limitations disappear. Thus we have: —

$$x_1 = a \sin x \ \mathrm{Sh}y, \qquad x_2 = a \ \mathrm{Sh}x \ \mathrm{Ch}y,$$

$$y_1 = b \cos x, \qquad y_2 = \pm\, b \ \mathrm{Ch}x,$$

$$z_1 = c \sin x \ \mathrm{Ch}y, \qquad z_2 = c \ \mathrm{Sh}x \ \mathrm{Sh}y,$$

and hence

$$-\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1 = -\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2}.$$

Imaginary values of the variables x and y in the equations would, however, make either set complete.

26°. Formula (11), for finding volumes, is

$$V = \tfrac{1}{3} \int_y \int_x \mathrm{S}\rho\rho'_1\rho'_2.$$

From the equation of the general ellipsoid, it is at once evident that

$$\mathrm{V}\rho\rho'_1 = \mathrm{V}\alpha\sigma, \ \mathrm{V}\sigma\sigma' = \mathrm{V}\beta\gamma,$$

whence follows

$$\mathrm{S}\rho\rho'_1\rho'_2 = \mathrm{S}\rho'_2\mathrm{V}\rho\rho'_1 = \sin x \ \mathrm{S}\sigma'\mathrm{V}\alpha\sigma$$

$$= \sin x \ \mathrm{S}\alpha\mathrm{V}\sigma\sigma' = \sin x \ \mathrm{S}\alpha\beta\gamma = abc \sin x.$$

The ellipsoidal volume is therefore

$$V = \frac{abc}{3} \int_y \int_x \sin x = \Phi \frac{abc}{3} [\cos x_0 - \cos x]. \qquad (52)$$

The whole volume is $V = \frac{4}{3} \partial abc$.

27°. From the equation of the parted hyperboloid are easily and directly obtained

$$\mathrm{S}\rho\rho'_1\rho'_2 = \mathrm{S}\rho'_2\mathrm{V}\rho\rho'_1 = \mathrm{Sh}x\mathrm{S}\sigma'\mathrm{V}\alpha\sigma$$

$$= \mathrm{Sh}x\mathrm{S}\alpha\mathrm{V}\sigma\sigma' = \mathrm{Sh}x\mathrm{S}\alpha\beta\gamma = abc\mathrm{Sh}x,$$

and

$$V = \frac{abc}{3}\int_y\int_x \mathrm{Sh}x = \Phi\,\frac{abc}{3}\Big[\mathrm{Ch}x\Big]_{x_0}^{x}. \qquad (53)$$

Similarly the equation of the unparted hyperboloid gives

$$S\rho\rho'_1\rho'_2 = -\,\mathrm{Ch}x\,S\sigma'\alpha\sigma = \mathrm{Ch}x\,S\alpha V\sigma'\sigma$$

$$= \mathrm{Ch}x\,S\alpha\gamma\beta = -\,abc\,\mathrm{Ch}x,$$

and

$$V = \frac{abc}{3}\int_y\int_x \mathrm{Ch}x = \Phi\,\frac{abc}{3}\Big[\mathrm{Sh}x\Big]_{x_0}^{x}. \qquad (54)$$

28°. Formulae (52), (53), and (54), it will be observed, give the volume of the portion of space swept by the radius vector; *i.e.*, of the space contained between the surface of the quadric and the two cones whose vertices are at the centre, and which are generated by the limiting positions of the radius vector at the limits of integration; or, if the lower limit is zero, the volume contained between the surface and a single central cone. The volume of the segment cut off by a plane perpendicular to an axis, can be found by finding the difference between the volume given by the general formula and the whole volume of this cone considered as limited by the plane; but more easily and simply by a change of origin, according to the method of 6°, by the use of formula (12).

29°. Beginning with the ellipsoid, let the origin be transferred to a fixed point on α, and let ϖ be the new generating vector. Then

$$\varpi = \rho - m\alpha,$$

m being a scalar. Formula (12) becomes in this case

$$V = \tfrac{1}{3}\int_y\int_x S\varpi\rho'_1\rho'_2 = \tfrac{1}{3}\int_x\int_y S(\rho - m\alpha)\rho'_1\rho'_2.$$

We have, as in 24°,

$$\rho'_1 = -\,\alpha\sin x + \sigma\cos x,$$

$$\rho'_2 = \sigma'\sin x,$$

hence

$$mS\alpha\rho'_1\rho'_2 = mS.\alpha(-\,\alpha\sin x + \sigma\cos x)\sigma'\sin x$$

$$= m\sin x\cos xS\alpha\beta\gamma = mabc\sin x\cos x.$$

In 26°, we found

$$S\rho\rho'_1\rho'_2 = abc\sin x.$$

The ellipsoidal volume will therefore be

$$V = \Phi\,\frac{abc}{3}\int_{x}(\sin x - m\sin x\cos x)$$

$$= \Phi\,\frac{abc}{3}\Big[\cos x + \frac{m}{2}\sin^2 x\Big]_{x}^{x_0}. \qquad (55)$$

If m be determined by the condition $m = \cos x$, where x is the superior limit of integration, then $\varpi = \sigma\sin x$ at that limit, showing that the volume obtained will be limited by a plane perpendicular to α; and if the lower limit is 0, we have for the volume of a segment: —

$$V = \Phi\,\frac{abc}{3}\Big[1 - \cos x - \tfrac{1}{2}\cos x\sin^2 x\Big]. \qquad (56)$$

30°. Similarly, the parted hyperboloid gives

$$\varrho'_1 = \alpha\mathrm{Sh}x + \sigma\mathrm{Ch}x,$$

$$\varrho'_2 = \sigma'\mathrm{Sh}x,$$

$$S\varrho\varrho'_1\varrho'_2 = abc\mathrm{Sh}x,$$

$$mS\alpha\varrho'_1\varrho'_2 = mS.\alpha(\alpha\mathrm{Sh}x + \sigma\mathrm{Ch}x)\sigma'\mathrm{Sh}x$$

$$= mabc\mathrm{Sh}x\mathrm{Ch}x;$$

and

$$S\varpi\varrho'_1\varrho'_2 = S\varrho\varrho'_1\varrho'_2 - mS\alpha\varrho'_1\varrho'_2,$$

the origin being changed to any assumed point on α; whence easily

$$V = \Phi\,\frac{abc}{3}\int_{x}(\mathrm{Sh}x - m\mathrm{Sh}x\mathrm{Ch}x)$$

$$= \Phi\,\frac{abc}{3}\Big[\mathrm{Ch}x - \frac{m}{2}\mathrm{Sh}^2x\Big]_{x_0}^{x}. \qquad (57)$$

Determining m by the condition $m = \mathrm{Ch}x$, where x is the superior and 0 the lower limit, this formula becomes

$$V = \Phi\,\frac{abc}{3}\Big[\mathrm{Ch}x - \tfrac{1}{2}\mathrm{Ch}x\,\mathrm{Sh}^2x - 1\Big]; \qquad (58)$$

and gives a portion of the *interior* volume, determined by a plane section perpendicular to α.

The unparted hyperboloid can be treated in a somewhat similar manner. In this case, we may assume

$$\varpi = \varrho - \sigma\mathrm{Ch}x = \alpha\mathrm{Sh}x,$$

which is equivalent to the condition that ϖ shall be the ordinate parallel to α for any assumed point of the surface. $\mathrm{S}\varpi\varrho_1{}'\varrho_2{}'dxdy$, being always the elementary parallelopiped four of whose edges are parallel and equal to $\alpha\mathrm{Sh}x$, will therefore represent the element of the volume which we are seeking. By easy transformations, we find

$$\mathrm{D}_x\mathrm{D}_y V = \mathrm{S}\varpi\varrho_1{}'\varrho_2{}' = \mathrm{Sh}^2x\mathrm{Ch}x\mathrm{S}\alpha\sigma\sigma'$$

$$= \mathrm{Sh}^2x\mathrm{Ch}x\mathrm{S}\alpha\beta\gamma = abc\,\mathrm{Sh}^2x\mathrm{Ch}x,$$

whence

$$V = abc\int_x\int_y \mathrm{Sh}^2x\mathrm{Ch}x = \Phi\,\frac{abc}{3}\Big[\,\mathrm{Sh}^3x\,\Big]_{x_0}^{x},$$

and if we take zero as the lower limit,

$$V = \Phi\,\frac{abc}{3}\Big[\,\mathrm{Sh}^3x\,\Big],$$

which gives an expression for the volume contained between the surface and the elliptical right cylinder whose axis is α and whose base is the section determined by the vector $\sigma\mathrm{Ch}x$.

Cambridge, June 1, 1877.

XXII.

ON A NEW METHOD FOR THE SEPARATION AND SUBSEQUENT TREATMENT OF PRECIPITATES IN CHEMICAL ANALYSIS.

By F. A. Gooch.

Presented Feb. 13, 1878.

The introduction of Bunsen's method of filtration and immediate ignition of precipitates in the moist condition has left little to be desired as regards accuracy of result and rapidity of execution, in the treatment of precipitates which may be submitted to high temperatures in contact with carbonaceous matter. In analytical methods which require that filter-paper and precipitate shall be ignited apart, or dried together at a temperature below the point at which paper begins to char, the same degree of exactness has not, in general, been hitherto attained.

To obviate the difficulty of bringing a paper-filter of ordinary dimensions, particularly when covered with a voluminous precipitate, to a definite condition of desiccation, the sand-filters of Dr. Gibbs and Taylor,* the porous cones of Munroe,† and finally the process of reverse filtering, first applied to quantitative work by Carmichael,‡ improved by Casamajor,§ and thoroughly elaborated by Professor Cooke,|| have been successively brought forward.

The latter process gives most excellent results in the separation of precipitates which settle quickly and completely; and, inasmuch as many precipitates which of themselves are not inclined to fall rapidly may be made to do so by proper treatment (by boiling, for example, the addition of ammonium salts, Chatard's ¶ method of granulating gelatinous precipitates, or other similar device), the field of usefulness of this process is wide.

The sand-filters in skilful hands, and the porous cones with no more than ordinary care, give accurate results, and possess moreover the

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* Am. Jour. [2], vol. xliv. p. 215.

† Am. Jour. [3], vol. i. p. 1.

‡ Zeitschr. f. Chem. [N.F.], Bd. 6, p. 481.

§ Am. Chem. vol. v. p. 441.

|| These Proceedings, vol. xii. p. 124.

¶ Am. Jour. [2], vol. l. p. 247.

advantage of being applicable to the filtration of liquids which quickly destroy woody fibre. The length of time required to dry them thoroughly, and their sensitiveness to atmospheric conditions, are their great drawbacks.

For the simple filtration of corrosive liquids without reference to a subsequent estimation of the precipitate, filters of sand, broken glass, garnets, and fibrous asbestus have for a long time been used. More recently, asbestus has been moulded into the shape of an ordinary filter: in the dry state by Löwe,[*] by rubbing between hollow and a solid wooden cones; in the wet state by Gruner,[†] by grinding asbestus, mixed in a mortar with water, to a pulp, transferring the mixture to a funnel choked with asbestus, inserting an accurately fitting cone of brass gauze, which presses the asbestus against the walls of the funnel, pouring off the water, carefully removing the cone and drying the layer of asbestus which adheres to the glass. Böttger [‡] has used filters of gun-cotton; and Bunsen [§] has devised a filtering apparatus for corrosive liquids to be attached to his pump, which consists of a disc of artificial pumice fitted to a conical tube and packed around its edge with fibrous asbestus.

None of these later-mentioned methods, however, are well adapted to the quantitative estimation of precipitates.

Impressed with the desirability of further improvement in those processes of quantitative analysis which involve the use of dried filters, or the separation of filter and precipitate before ignition, I have had the good fortune, in taking the matter up in turn, to succeed in devising and preparing a felt of anhydrous asbestus, which is capable of filtering liquids with a rapidity and efficiency at least as great as may be obtained by the use of good filter paper; is light, compact, incombustible at the highest temperatures used in analytical processes; is not acted upon by acids (excepting hydrofluoric acid) or alkalies; is sufficiently coherent to resist entirely the disintegrating action of a liquid forced through it under the pressure of the Bunsen pump, and which may moreover be prepared by a very simple process: in short, a filtering material which, in my belief, makes it possible to reach a high degree of accuracy in many analytical processes which hitherto have been none of the best, and to add to those already known new methods which previously have been impracticable.

My mode of preparing and using the asbestus felt is as follows: —

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[*] Dingl. pol. Jour. cxlviii. 444.

[†] Jahresb. Chem. 1869, p. 990.

[‡] Dingl. pol. Jour. clv. p. 463.

[§] Ann. Ch. Pharm. cxlviii. p. 290.

First, white, silky, anhydrous asbestus is scraped to a fine short down with an ordinary knife-blade, boiled with hydrochloric acid to remove traces of iron or other soluble matter, washed by decantation and set aside for use.

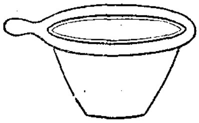
Fig. 1.

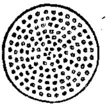
Fig. 2.

Secondly, a platinum crucible of ordinary size, preferably of the broad low pattern (Fig. 1), is chosen, and the bottom (Fig. 2) perforated with fine holes (the more numerous and the finer, the better) by means of a steel point; or, better still, the bottom may be made of fine platinum gauze. Next, a Bunsen funnel of the proper size is selected, and — following Munroe's plan for holding his porous cones — over the top a short piece of rubber tubing is stretched and drawn down until the portion above the funnel arranges itself at right angles to the direction of the stem. Within the opening in the rubber, the perforated crucible is fitted as shown in figure 3, and the funnel is connected with the receiver of a Bunsen pump or other exhausting apparatus in the ordinary manner.

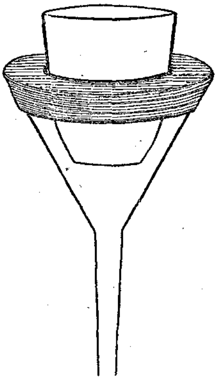
Fig. 3.

To make the asbestus felt, the pressure of the pump is applied, and a little of the asbestus prepared as described, and suspended in water, is poured into the crucible. The rubber and the crucible are held together by the pressure of the vacuum-pump with sufficient force to make an air-tight joint; the water is drawn through, and the asbestus is deposited almost instantly in a close compact layer upon the perforated bottom; more asbestus (if necessary) in suspension as before being poured upon the first, until the layer becomes sufficiently thick for the purpose for which it is intended. Finally, a little distilled water is drawn through the apparatus to wash away any filaments that might cling to the under side, and the filter is ready for use: the whole process occupying less time than is necessary to fold and fit an ordinary paper-filter to a funnel.

To prepare the filter for the estimation of a precipitate, the crucible with the felt undisturbed is removed from the funnel and ignited. In case the precipitate, to be subsequently collected, must be heated to a very high temperature for a long time, it is better to enclose the perforated crucible with its felt within another crucible; because, in such cases, asbestus felt is apt to curl at the edges, and without such precaution some of the precipitate might drop through the perforations and be lost. For drying at low temperatures, however, and even for ordinary ignitions, a second crucible is unnecessary; but, during the ignition of an easily reducible substance, care must be taken to prevent the contact of unburnt gas with the perforated bottom.

To perform the filtration, the crucible is replaced in the funnel, the pressure applied, and the process conducted precisely as in an ordinary filtration by the Bunsen pump. It is necessary to observe that the vacuum-pump is to be started before pouring the liquid upon the filter. The final drying or ignition, as the case may be, of precipitate and filter is made without difficulty, or need of extra precaution.

When turbid liquids are to be filtered, or gelatinous precipitates to be separated, instead of the perforated crucible, I prefer to use a platinum cone (Figs. 4 and 5), the upper part of foil (to make a tight joint with the rubber fitting of the funnel), the lower of gauze. The method of covering the gauze with felt is identical with that described above. By reason of the larger filtering surface of this apparatus, the tendency to become clogged is, of course, very much diminished. When subjected to prolonged ignition, the gauze cone is enclosed within a crucible or a cone of platinum foil.

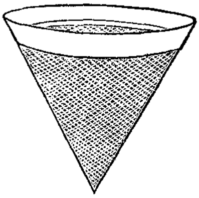
Fig. 4.

In operations in which platinum is liable to receive injury, a porcelain crucible, with a perforated bottom, may be used; but recourse to this is rarely necessary, particularly when one may use the gauze cone protected, as it is, by asbestus felt; moreover, the perforation of porcelain with numerous fine holes is a matter of considerable difficulty and expense.

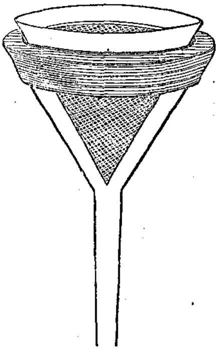
Fig. 5.

Asbestus felt may be also used in the process of reverse filtering, it being merely necessary to dip the platinum rose into the asbestus mixture, after starting the vacuum-pump, in order to make the felt. The rose, with the felt attached, and the vessel in which the precipitate is collected, are to be weighed together, both before and after filtration.

Nothing can be simpler than the whole method of preparation and use of the apparatus which I have described, and its efficiency is extremely great. Clean water may, under the pressure of a Bunsen pump, be passed through a gauze cone coated with asbestus felt, which exposes a filtering surface of twenty-four square centimetres (nearly the same as that of a paper-filter, eight centimetres in diameter, when folded in the ordinary manner), with ease at the rate of a litre per minute.

When the filtering surface is less, the rapidity of filtration is, of course, somewhat diminished, but always exceeds that of paper of the same dimensions. When the felt is deposited upon gauze, the layer may be surprisingly thin and yet be efficient enough for all ordinary purposes. If the layer of felt be quite thick, the filtrate from baric sulphate freshly precipitated in the cold, may be made to pass through clear. But the great superiority of asbestus felt lies in its constancy of weight, whether dried at high or low temperatures, the rapidity with which it may be safely and completely dried, and its refractoriness as regards acids (excepting hydrofluoric acid) and alkalies. These advantages appear most prominently in processes which involve the separation and desiccation: —

First, of precipitates which (like K_2PtCl_6, $[Hg_2]Cl_2$) must be dried at low temperatures, on account of a tendency to decompose spontaneously at high temperatures;

Secondly, of precipitates which (like Sb_2S_3, $PbSO_4$, $ZnCO_3$) have usually been dried slowly and tediously at low temperatures, or have been separated from the filter before ignition, to avoid the danger of reduction by heating in contact with carbon;

Thirdly, of precipitates which (like $(NH_4)_2Mg_2P_2O_8$, $(NH_4)_2Mn_2P_2O_8$) may be ignited in contact with carbon, but which make its complete combustion difficult.

In all cases, the time required to dry or ignite precipitate and filter is a minimum, inasmuch as heat may be applied as soon and to as high a degree as the precipitate itself will permit. Even in processes in which paper filters find their most convenient application, paper has generally no superiority over asbestus felt.

To illustrate the cohesiveness and refractory nature of the felt, I append the account of a single experiment. All of the liquids used had been previously filtered, excepting the distilled water with which the filter was washed, after the passage of each liquid. This precaution was considered unnecessary in the case of the water; but, if it had been taken, the slight *increase* of weight which was observed during the experiment would probably have been prevented.

Weight, in grammes, of crucible and felt, after ignition	20.2020
after second ignition	20.2020
after passing 100 cm^3 of strong HCl through the filter *ten* times, washing and igniting	20.2020
after passing 100 cm^3 of strong HNO_3 through the filter *ten* times, washing and igniting	20.2021
after passing 100 cm^3 of strong H_2SO_4 through the filter *five* times, washing and igniting	20.2022
after passing 100 cm^3 of a mixture of strong H_2SO_4 and water in equal parts, *ten* times through the filter, washing and igniting	20.2024
after passing 100 cm^3 of water, containing in solution 50 grammes of $NaOH$ *ten* times through the filter, washing and igniting	20.2028
Weight of crucible	20.1932
„ „ felt	.0088

The whole process, involving, in all, the filtration of five litres of liquid (of which one litre was viscous), more than forty changes of the receiver, together with the ignition, cooling, and weighing of the crucible and felt seven times, was completed in two hours and ten minutes. The disc of felt was eleven millimetres in diameter.

XXIII.

ON PEIRCE'S CRITERION.

By Benjamin Peirce.

My dear Sir, — I perceive that the theory of my criterion has been frequently misunderstood. I presume this to be due in a great degree to the conciseness of the argument with which it was published; and I propose to remedy this defect.

The problem which I undertook to solve was the following. There being given certain observations, of which the greater portion is to be regarded as normal and subject to the ordinary law of error adopted in the method of least squares, while a smaller unknown portion is abnormal and subject to some obscure source of error, to ascertain the most probable hypothesis as to the partition of the observations into normal and abnormal. The principle adopted in my solution of the problem is the universally recognized doctrine that the measure of the probability of an hypothesis compared with other hypotheses equally probable in other respects is the probability that the event will occur under the hypothesis, and that the most probable hypothesis is that under which the event is most probable. This is the literal expression of the mathematical analysis published in Gould's Astronomical Journal for 1852. The Criterion has been used otherwise than in the Coast Survey, and especially by my friend Dr. Gould himself. Dr. Gould's tables have greatly facilitated its use, and his sound judgment given in favor of its validity is at least as valuable as that of any living geometer. It has also been much used by that excellent authority Mr. Schott, as in a letter hereto appended.

The evidence, by which certain observations are placed in the doubtful list and subjected to scrutiny, whether they should be rejected, must be exclusively the magnitude of the errors which they involve, when these errors are computed as if they were normal observations. This would not seem to be obnoxious to the charge of inconsistency, any more than is the ordinary *Reductio ad Absurdum,* in using a method as correct in an observation where it was finally rejected. An hypothesis which leads to improbable results is itself improbable to just that extent.

It is not likely that a series of observations of any great extent was ever made, in which some were not rejected on account of the magnitude of the deviations from the arithmetic mean. The object of the Criterion is to systematize this species of rejection, and reduce it to a form of exact computation. Wherever it has been applied, I believe that it has been found to accord with the best judgment of observers and computers. This fact has been to me an agreeable surprise, which has more than balanced the unfavorable criticism, having its foundation in misconception. It has been a surprise, because the theory was altogether *a priori*, and independent of comparison with observation.

I might add concerning the charge of inconsistency, that it would seem to be almost equally applicable to the case where two sets of observations made by different observers, and the arithmetical mean of each set differing materially from the arithmetical mean of the whole, the difference is explained by personal equation. The argument for the hypothesis of personal equation must rigidly assume the same form with that by which the Criterion is established.

There might be a doubt as to the reality of such abnormal observations with their obscure sources of error. I am frank to admit that in most cases science will detect the system of the sources of error, and free the observations from them. But even such a case as the familiar one of writing down a wrong figure must be included among these sources, and is evidently insusceptible of correction; whereas, if it be sufficiently large, it will be eliminated by the Criterion. Another case, which must often have occurred in transit observations by eye and ear, depends upon the erroneous mental sub-division of the record, of which I have given the analysis in another place. I can point out many observations which were manifestly erroneous on this account, and which it would be too vast a labor to undertake to rectify. Before the judgment of the Criterion all such errors disappear, if they are large enough to be of serious injury.

That some of the observations which are not rejected may be abnormal, notwithstanding the smallness of their errors, must be admitted. This possibility was fully recognized in the geometrical development which was given in the Astronomical Journal; and I am not aware that there has been any criticism adverse to the mathematics of that article.

To Hon. C. P. Patterson,
Superintendent U. S. Coast Survey.

NOVEMBER 22, 1877.

Respecting Prof. Peirce's Criterion, I venture a few remarks: —

Large errors may arise either from an *accumulation* of a number of small ones, having their origin in different sources, as is recognized in the theory of combination of errors, and as such they may be regarded as *normal* (even if quite large), provided they belong to a series of an indefinitely large number of observations. Practically, we have but a few observations (most frequently less than one hundred), and while certain sources of error may combine to the production of large ones, their *actual* appearance in a *short* series of observations must injuriously affect the most probable result (say the mean) deducible from the series. Here we need the means of separation, and the Criterion logically performs this.

Or large errors may arise from bad observations (due to inattention of observer, without being aware of it), from the presence in this particular case of an unsuspected constant error, or even from an accidental slip (necessarily not altogether outside the possibility of its being due to other causes admissible): all such large errors having no recognized place in the adopted law of the occurrence of error must be subject to rejection, for which we need the Criterion.

Supposing the probable error of a result before rejection $= \varepsilon$ and after rejection $= \varepsilon_1$, the latter value has some times been adopted, which I think is generally not admissible, the value ε would better be retained as evidence that rejections have been made. If ε_1 is retained, we may fall into the error, of again trying on the Criterion* based upon ε_1.

Having used the Criterion for the last twenty years in various investigations, I found it uniformly gave excellent discrimination, and do not remember a single case where it came in conflict with proper judgment based upon experience. Of course in those instances where we have antecedently a knowledge of ε, we would employ this value in preference to a value deduced from a *single* series of observations: in such cases, observations are frequently retained by the Criterion which otherwise would have been rejected. If it errs at all, it may sometimes appear to cut too deep; but our general experience is that probable errors deduced by the method of least squares generally prove too small, whenever we can bring them to the test of different methods: this, of course, is due to the presence of unknown constant errors peculiar to each method.

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* I remember a rather curious case of this kind, where an observer rejected successively three times, each time deducing and starting from a new probable error, when he became alarmed for the safety of the rest of his observations and stopped further testing them.

Yours very respectfully,
CHAS. A. SCHOTT.

CARLILE P. PATTERSON, Esq.,
Superintendent U. S. Coast Survey.

XXIV.

NOTE ON THE MEASUREMENT OF SHORT LENGTHS.

BY LEONARD WALDO.

Assistant at Harvard College Observatory.

IT is often desirable in practical astronomy to determine short linear units with such a degree of accuracy that the errors in the unit may be disregarded, in comparison with the errors of the observations in which it is involved. Such instances as the determination of the errors of micrometer screws, the single divisions of large circles, the apertures of diaphragms and ring-micrometers, the intervals between micrometer threads, may be readily cited, in which tedious numerical computations and considerable observing would be avoided, if such units could be readily submitted to an investigation under the very high magnifying power of the microscope relative to an eye-piece.

In the usual method of comparing short lengths with the microscope by means of an eye-piece micrometer, we meet the difficulty that no greater distance can be measured at one operation than can be included within the two extreme lines of the micrometer in the field of view. In this case, resort must often be had to low-power objectives, in which event the micrometer may include a desired space beyond the field of a higher power; but, at the best, the microscope eye-piece micrometer fails in all cases where so long consecutive distances as 0.1 inch are to be measured. The expense of the exquisite comparators made by Repsold, Froment, Brunner Frère, and Troughton and Simms, places them beyond ordinary reach. And the current idea that exact measures must be made with the aid of arbitrary scales, whose divisions may always be assumed to be relatively the same, is apt to cause us to overlook the extreme precision now attained in the construction of short screws, and the methods of measuring adapted to the stage of the microscope.

The screw stage micrometer suggested itself as an available way of submitting short linear units to exact measurement, provided the stand of the microscope be made of greater stability than in usual constructions, and that the screw itself be of accurate workmanship.

It is not material in such measurements that the zero of the scale should retain a fixed position for more than a few hours together. The screw is so short that it most probably is affected throughout its length by the same conditions of temperature and thickening of the oil. And with the micrometer screw we can apply the well-known principle that a bisection of a small object can be made more exactly than can the distance of that object be estimated relative to two micrometer lines contiguous; unless, indeed, the object is placed midway between two closely parallel lines, which becomes then also a case of bisection.

In order to carry out the idea, Mr. Crouch constructed for me one of his first-class microscope stands, with some modifications in it which I thought necessary to insure the solidity we find in astronomical instruments. A clamp is added to the axis on which the instrument swings, so that it may be rigid at any inclination. The "Jackson arm" contains a small clamp so that any possible play in the rack and pinion can be counteracted. In the Crouch model, this arm has a bearing of 17 cm. in length, and 16 mm. in width.

Resting upon this bearing is the cradle which carries the body of the microscope; its base is 16 mm. in width, and the chord of its upper circular surface is 19 mm. The body, which is constructed of brass tubing, 2 mm. thick, and 36 mm. interior diameter, is soldered to this cradle. The side of the cradle away from the body carries the ordinary T rail with a smooth-working rack. The pinion is provided with large heads, 5.7 cm. in diameter, and the performance is satisfactory enough to readily focus a high angle sixth upon an object, without resort to the fine adjustment which, in the Jackson model, unfortunately alters the distance between the object-glass and the reticule in the eye-piece.

The screw and pinion moving the mechanical stage are provided with large heads 3.7 cm. in diameter, for the purpose of more easily re-setting upon the first line of a series in measuring the same space with different parts of the micrometer screw to be hereafter mentioned. The ordinary triple-threaded screw for carrying the mechanical stage being too coarse to allow of exact setting, Messrs. Buff & Berger have replaced it for me with a screw having forty-one threads to the English inch. This screw is opposed to the micrometer screw, so that the principle of repetition may be used in measures where two contiguous lines of a scale appear in the same field of view.

The fine adjustment is provided with an unusually stiff spring, to avoid possible change when once set. The eye-pieces are provided with close-fitting collars, so that the draw tube may be removed and the eye-pieces inserted directly into the body of the microscope.

The micrometer stage was constructed by G. & S. Merz, of Munich. It is their screw micrometer stage, adapted originally to their own microscope stands. It consists essentially of a slide moving upon a base plate 75 × 77 mm., and between two ledges adjusted with sides parallel to the slide and the axis of the micrometer screw. The slide in section is symmetrical, with its upper edge 3.35 cm., its lower edge 3.90 cm., and its vertical 5.0 mm.; its upper surface has a length of 7.4 cm. The slide carries upon its upper surface another slide, which by a rack and pinion is moved at right angles to the axis of the screw. This motion is necessary in order to assure an observer that lines of the series he may be about to observe are placed at right angles to the axis of the screw.

The slide first mentioned is pulled by spiral springs with a force varying from 0.7 kil. to 2.0 kil. against the end of the screw, as the screw moves the slide from one end of its run to the other, the bearing surface being of steel. The nut through which the screw turns is fixed to the lower plate on which the slide moves. This nut may be adjusted for position, *i.e.* to render it concentric with the screw, and its friction on the screw may be altered by turning a small screw which passes through the nut on one side. This side has been cut through, so that the small screw has really the nature of a clamp screw.

The sliding plate carries a pointer indicating whole revolutions of the screw on a silvered scale fixed to the lower plate.

The screw itself is of steel, and it is cut as nearly as practicable with 75 lines to the Paris inch. It is cut over a length of 26 mm., and is 3.7 mm. in diameter at the bottom of the screw spiral. It has the ordinary pattern micrometer head 46 mm. in diameter, which is divided into 100 parts, each of which may be subdivided into 20 parts, or even to a less degree by estimation by means of a mica scale and a small magnifying lens. The nut is of red metal, and has an upper surface rectangular in shape with a breadth of 14 mm. and a length in the direction of the screw axis of 11.1 mm., thus preserving a ratio of 3 : 1 with reference to the diameter of the screw.

It might be remarked that this ratio is an old established one; but that Mr. Adam Hilger tells me he has lately constructed some small screws, in which the relation of the nut to the diameter of the screw was disregarded, but the nut was constructed $\frac{3}{4}$ the length of the screw. He spoke highly of his success with this construction.

The screw in use is slightly oiled with an unguent consisting of equal parts of beeswax and tallow, with about $\frac{1}{30}$ part of clock oil added.

To facilitate exact setting with the screw, a smoothly turned and thin wooden disc 8.5 cm. in diameter slips over the screw head, to be clasped at its opposite edges by the fingers and thumb, in turning the screw. The whole micrometer screws to a stage plate, which may be readily slipped into the grooves cut in the stage of the microscope stand ordinarily to receive the object-holders.

The results given of the measures of short standards by this apparatus would be of little interest, unless accompanied by the results of an investigation of the errors to which a single setting of the screw is liable.

A simple method of investigating at once the errors depending upon the graduation of the head of the screw, of the variation in different parts of the same revolution, and of any cumulative error in the length of one revolution at different distances from the assumed zero of the scale, is to use a single band in the manner described below of the width of the value of one revolution, consisting of as many lines ruled on glass as there are units in the denominator of the fraction expressing the value of the smallest fractional part of the head to be considered.

The first line of this band, when the whole band has been passed over, is brought successively back to the index in the eye-piece, which should be perhaps two parallel lines nearly the same distance apart as the apparent width of the line on the stage micrometer as seen in the field of view. One of the screws of the mechanical stage is used for this purpose. This band should have lines enough upon it to have two consecutive ones in the field at once with a high power objective, in order to have its errors investigated with an eye-piece micrometer, and independently of the screw. It should be borne in mind that in this case the measures should be made in the same part of the field, to avoid errors arising from the unequal distortion of the eye-piece lenses. We thus avoid the otherwise necessary examination of a long scale of lines; and it is my opinion that it is safer to make the more numerous settings required by this method, than to trust to any inexhaustive treatment of a series of many lines, such as would be necessary without a considerable expenditure of time.

In determining the mean value of one revolution, we shall derive an advantage in using the mean of the ten settings for terminal points.

If now we put

$d =$ the number of spaces in the band,
$\gamma_0 =$ the micrometer reading on the first line,
$\gamma_1 =$ " " " " second line,
$\gamma_n =$ " " " " n^{th} line,
$M_0 =$ the mean reading of the first d lines,
$M_1 =$ " " " " second d lines,
$M_n =$ " " " " $(n + 1)^{th}$ d lines,
$m_p =$ the value of 1 rev. at p revolutions,

we have

$$m_p = M_{p+1} - M_p$$

when the spaces in the band are commensurate with the value of one revolution.

We have also the accumulated error, from the p^{th} to the $(p + a)^{th}$ term, —

$$E = (m - m_p) + (m - m_{p+1}) \; . \; . \; . \; . \; (m - m_{p+a}),$$

depending on the whole revolutions.

The value of the corrections to be applied depending on the irregularities of single parts of one revolution, will be of the general form : —

$$e = \frac{m_p}{d} - \left\{ \gamma_{(pd - \frac{d}{2} + 1)} - \gamma_{(pd - \frac{d}{2})} \right\}$$

In the present case we have assumed $d = 10$, and the value of the screw is investigated for each $\frac{1}{10}$ of a revolution from $0.^{r}00$ to $25.^{r}0$ of the scale.

I am indebted to the courtesy of Prof. W. A. Rogers for a band of ten lines corresponding at 75° F. to one revolution of my screw, and so equably spaced that the spaces are sensibly uniform with any powers used in the following investigation. I should readily have detected a difference so great as 0.00001 of an inch between any two spaces of the series.

In making the observations from which the following results are derived, a $\frac{1}{5}$ objective by Crouch, having an angular aperture of 100°, and adjusted for glass cover of the slide, was combined with a short-focus negative eye-piece provided with a reticule on cover glass placed in the focus of the eye-lens. The magnifying power was 1050 diameters, nearly.

For illumination, the edge of a flame of a kerosene lamp was placed in the focus of a system of condensers 4½ inches in diameter, and a beam of rays was thrown from the distance of three feet upon the concave mirror, which reflected them centrally upon the glass plate containing the band. It was found that this illumination answered the purpose; for though the lines did not show the detail visible with monochromatic light and sub-stage condenser, yet, being comparatively widely separated, they were well adapted to measurement, when lines ruled closely would have been measured with difficulty.

The following table contains the results in millimeters of the investigation relative to periodicity of the values of one revolution. M_0 being the mean of the first ten readings of the screw, and E being the sum of the residuals of $m_{0 \text{ to } 24}$ from the mean value of one revolution.

	$M_{0 \text{ to } 24}$.	$m_{0 \text{ to } 24}$.	$m - m_{0 \text{ to } 24}$.	E.	E in mm.
	r.	r.			
M 0	0 54856	.9986	—.0002	—.0002	—.00007
1	1.54717	.9970	+.0014	+.0012	+.00043
2	2.54418	.9974	+.0010	+.0022	+.00079
3	3.54153	.9975	+.0009	+.0031	+.00112
4	4.53904	.9981	+.0003	+.0034	+.00128
5	5.53712	.9980	+.0004	+.0038	+.00137
6	6.53515	.9977	+.0007	+.0045	+.00162
7	7.53384	.9995	—.0011	+.0034	+.00123
8	8.53338	.9937	—.0013	+.0021	+.00076
9	9.53306	.9987	—.0003	+.0018	+.00065
10	10.53173	.9978	+.0006	+.0024	+.00087
11	11.52950	.9994	—.0010	+.0014	+.00051
12	12.52894	.9994	—.0010	+.0004	+.00014
13	13.52798	.9986	—.0002	+.0002	+.00007
14	14.52656	.9995	—.0011	—.0009	—.00032
15	15.52608	.9989	—.0005	—.0014	—.00051
16	16.52496	.9987	—.0003	—.0017	—.00061
17	17.52363	.9935	—.0001	—.0018	—.00065
18	18.52212	.9972	+.0012	—.0006	—.00022
19	19.51934	.9978	+.0006	—.0000	—.00000
20	20.51713	.9989	—.0005	—.0005	—.00018
21	21.51561	.9975	+.0009	+.0004	+ 00014
22	22.51306	.9994	—.0010	—.0006	—.00022
23	23.51241	.9981	+.0003	—.0003	—.00011
M 24	24.51046				

Mean value of $m = \overset{\text{rev.}}{0.99841}$

The following table contains the residuals of the separate readings from the mean value of each revolution, also expressed in millimeters: —

VALUES OF *e* IN PARTS OF A MILLIMETER.

Rev.	.0 to .1	.1 to .2	.2 to .3	.3 to .4	.4 to .5	.5 to .6	.6 to .7	.7 to .8	.8 to .9	9 to 1.0
0	+.00000	+.00002	—.00003	+.00003	— 00004	+.00003	—.00002	+.00000	+.00000	—.00001
1	—.00002	+.00001	—.00008	+.00004	+.00002	+.00000	+.00004	—.00004	—.00001	+ 00003
2	—.00002	+.00004	—.00004	+.00005	+.00000	—.00001	+.00002	—.00009	+.00005	+.00001
3	+.00002	+.00000	—.00006	+.00005	+.00001	—.00001	+.00005	—.00007	+.00002	— 00001
4	+.00000	+.00001	—.00001	+.00003	+.00001	—.00004	+.00003	—.00001	—.00001	+.00001
5	+.00001	+.00002	—.00002	—.00000	+.00003	—.00003	+.00003	—.00001	—.00003	+.00000
6	+.00005	—.00002	+.00000	—.00001	—.00000	—.00002	+.00002	—.00001	—.00001	—.00002
7	— 00002	+ 00004	—.00002	+.00001	+.00002	—.00000	+.00003	—.00003	—.00001	+.00001
8	—.00001	—.00000	—.00003	+.00003	+.00001	+.00002	—.00000	—.00000	+.00000	—.00000
9	+.00001	+.00000	—.00005	+.00004	+.00000	+.00002	+.00000	+.00000	— 00001	—.00001
10	+.00000	—.00000	—.00001	+.00001	—.00001	+.00000	+.00003	—.00005	+.00001	+.00000
11	+ 00002	+.00003	—.00000	—.00003	+.00002	+.00001	—.00001	+.00001	—.00000	—.00004
12	+.00001	+.00000	—.00001	+.00003	+.00000	+.00002	+.00001	+.00001	—.00003	—.00002
13	—.00001	+.00001	+.00001	+.00001	—.00002	+.00003	—.00001	+.00002	—.00003	— 00001
14	—.00000	+.00006	—.00003	+ 00001	+.00000	—.00000	+.00004	—.00004	—.00002	—.00000
15	—.00001	—.00001	—.00002	+.00002	+.00000	+.00002	+.00000	+.00002	—.00004	+.00001
16	—.00001	+.00001	—.00001	+.00001	—.00000	+.00005	—.00002	+.00003	—.00005	—.00000
17	+.00001	+.00002	+.00001	—.00001	—.00000	—.00001	+.00003	—.00002	—.00002	+ 00001
18	+.00002	+.00003	—.00002	+.00002	—.00002	—.00001	+.00004	—.00002	—.00004	+.00002
19	+.00000	+.00003	—.00004	—.00001	+.00002	+.00001	+.00002	+.00002	—.00005	+.00000
20	+.00003	—.00001	—.00001	+.00001	—.00003	+.00003	—.00001	—.00000	+.00000	—.00001
21	+.00000	+.00002	—.00003	+.00000	+.00001	+.00003	—.00002	+.00003	—.00000	—.00004
22	+.00004	—.00003	+.00002	—.00002	—.00001	+.00001	—.00000	—.00001	—.00002	+.00001
23	—.00005	—.00001	+.00000	+.00003	+.00000	+.00003	—.00001	+.00001	—.00001	+.00001
24	—.00001	+.00000	—.00001	+.00003	—.00002	+.00003	+.00001	—.00000	—.00000	—.00004

e in no case being so great as a unit in the fourth decimal place.

I think from the above results that we are not warranted in assigning any error of eccentricity in the screw-head, or of sensible variation in value of the different parts of the single revolutions.

There is a sensible periodic error depending upon the entire number of revolutions. This periodic error is probably a function of the pressure exerted by the springs. It is not the purpose of the present paper to discuss the absolute errors of the screw, but simply to point out their probable amount at arbitrary intervals.

In this screw, as in all screws adapted to exact measurement, it is preferable in comparing two lengths to set the screw-head at the same zero for the first line in each of the two lengths; and if the measures are made in the centre of the field the distortion of the microscope lenses is insensible.

If an eye-piece micrometer is used, it is necessary that all measures be made in the same part of the field. And if that much more exact instrument (in the writer's opinion), the filar micrometer, be applied to the eye-piece of such a microscope comparator as described above, any measure within the field will be executed with the extreme of precision. The errors of the eye-piece micrometer screw are, in this case, approximately multiplied by one-tenth of the focal length of the objective. It is necessary, however, to take the same precautions as with the eye-piece micrometer, in regard to using the same part of the field. It is also better to begin with the same zero of the micrometer head in consecutive measures, and use the same part of the screw; though of course it is not so important here as in the case of the screw stage micrometer.

The preceding remarks are based upon the following considerations relating to the distance between two lines which are seen in different parts of the field of view at the same time:—

1°. A distortion of this distance may be caused by the objective, or the eye-piece, or both.

2°. The lines of an eye-piece filar micrometer may be so illuminated that the apparent distance between two lines in the field of view is not truly measured in bisecting first one and then the other.*

3°. The filar micrometer has errors of its own screw which are variable for different parts of its length, but which probably are sensibly the same for the same interval repeatedly used within a short time.

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* Professor Newcomb, in his paper on the Uranian and (Neptunian) systems, Washington Astr. Obs. for 1873, points out that this source of error may be remedied by using an achromatic eye-piece. It can only occur when the micrometer lines and the object measured are apparently of different colors.

To determine the value of one revolution of the screw of the stage micrometer, and more particularly the ratio existing between the various short standards available, I give the following measures: —

Date.	Temp. F.	Object Measured.	Resulting value of 1 revol'n.	Limiting Readings of Microm. Scale.	No. of Settings.	Reduced to 70° F.
			mm.	r. r.		mm.
1877. Sept. 28	77°	mm. on cover-glass by Froment, of Paris .	0.36072	8.1 to 10.9	6	0.36072
,, July 24	77°.5	* the same	0.36075	9.8 to 12.6	6	0.36075
,, Dec. 7	67°.8	the same	0.360685	Mean of 3.0 to 5.8 15.5 to 18.3 and 25.5 to 28.3	12	0.360684
,, Sept. 28	77°	mm. on cover-glass *marked* "Secretan," Paris	0.36111	8.1 to 10.9	6	0.36111
,, July 24	77°.5	* the same	0.36114	9.8 to 12.6	6	0.36114
,, Dec. 7	67°.8	the same	0.361180	3.0 to 5.8 15.0 to 17.8 25.7 to 28.5	12	0.361179
,, Dec. 7	67°.2	Lines on glass plate compared with U. S. C. S. standard centimeter	0.361330	1.0 to 36.9	6	0.361329
,, Dec. 7	67°.3	Lines on glass plate compared with U. S. C. S. "Brunner Frère" standard centimeter	0.36121	1.0 to 28.8	8	0.36121
1878. Feb. 2	63°.0	Electrotype copy of the U. S. C. S. "Brunner Frère" standard centimeter	0.361125	3.0 to 30.7	12	0.361134

* Represented to be copies at 15° C. of the standards of the International Bureau of Weights and Measures, at Paris.

The 5th column contains the limits within which the screw readings were confined, and the last column is computed by assuming

the coefficient of expansion of white glass to be 0.00 00 086 for 1° C.
,, ,, ,, untempered steel ,, 0.00 00 108 ,,
and ,, ,, ,, copper ,, 0.00 00 172 ,,

and applying the small differential corrections to the results in the fourth column.

Whence the successive standards, expressed in terms of the first as unity, are

mm. Froment = 1.00000
mm. Secretan = 1.00119
U. S. C. S. cm. = 1.00172
Brunner Frère cm. = 1.00139
Copy ,, ,, = 1.00117

I wish to thank Mr. R. W. Willson of Harvard College for his skilful aid in making the micrometer readings necessary in the first and second tables.

XXV.

CONTRIBUTIONS TO THE BOTANY OF NORTH AMERICA.

By Asa Gray.

Presented Jan. 9, 1878.

1. *Elatines Americanæ.*

§ 1. Crypta, Seubert. Isostemones (di–triandræ), oppositifoliæ: capsulæ dissepimenta tenuia nunc evanida. Flores in Americanis semper sessiles, trimeri nunc disepali: semina leviter curvata.

1. E. triandra, Schkuhr. Folia oblanceolata vel oblongo-lanceolata, basi sensim attenuata: petala stamina et carpella sæpissime 3 cum sepalis 2: semina fere subsequentis vel tenuiora, minus insculpta.

2. E. Americana, Arn. Folia obovata obtusissima: flores sæpius dimeri, nunc trimeri: semina cylindracea, curvula, lin. circiter $\frac{1}{3}$ longa, testa in lineis 9–10 multi-(20–30-) clathrata.

3. E. brachysperma, n. sp. Folia oblonga seu ovalia basi attenuata, nunc sublanceolata: flores plerumque dimeri: semina brevi-oblonga rectiuscula, haud ultra lin. $\frac{1}{4}$ longa, testa in lineis 6–7 grossius 10–12-clathrata.

§ 2. Elatinella, Seubert. Diplostemones (tetrameræ octandræ, rarissime trimeræ hexandræ), oppositifoliæ.

4. E. Californica, n. sp. Folia obovata basi longius attenuata, inferiores in petiolum lamina haud longiorem: flores breviter pedunculati: semina circinato-incurvata *E. Hydropiperis.*

My attention was recently called to our species of this genus by a letter from Mr. James Lloyd, communicating specimens of an *Elatine* which he found growing abundantly in the vicinity of Nantes, France, which he wished to have critically compared with our *E. Americana,* but which has, as he remarked, much narrower leaves and trimerous flowers, except that the sepals are almost always dimerous. If new, Mr. Lloyd proposed to name this plant *E. inaperta;* because that, while the flowers of all the European species are open or expanding, in this and in *E. Americana*, as far as he had seen, they are always closed. I cannot well distinguish the specimens sent from Swedish ones of *E. triandra*, which equally appear not to have expanded their blossoms; and from a remark of Seubert's (Elat. Monogr. 54), and from his reference to one by Braun, it may be inferred that this occurs, more or less, in the submersed state of *E. hexandra*, DC., his *E. paludosa.* In our American species, we have both closed and open blossoms, at least in *E. Americana;* and the petals in some terrestrial states of the latter are so large, conspicuous, and enduring, and so strikingly tinged with pink or rose-color, that they would seem to belong to a totally distinct species. The more aquatic forms, even when flowering above the water, probably expand transiently if at all, and have the appearance of being close-fertilized in the bud. But observations as to this should be made upon living plants.

The French specimens sent by Mr. Lloyd at once recalled the narrow-leaved plants of the Western Atlantic States, which had been unwittingly referred to *E. Americana*, regarded as the only N. American species; and on comparison they seem to be identical. I find, moreover, that Seubert has identified Chilian specimens, collected by Bertero, with *E. triandra.* [In a pamphlet on the Flora of the West of France, dated Dec. 30, 1877, Mr. Lloyd has published a detailed description and account of his species, under the name of *Elatine inaperta.*]

Among the herbarium specimens inadvertently referred to *E. Americana*, I find several, of more or less terrestrial habit, with leaves intermediate in form between those of *E. Americana* and of *E. triandra*, and with seeds distinct from either. I distinguish this as a species under the name of *E. brachysperma.*

Finally, among the many interesting plants received from the sharp-sighted and enthusiastic Mr. Lemmon, I find an *Elatine* of the *Elatinella* section, which was not before known in America.

So that, instead of a single species, we can now recognize four, the characters of which are presented in the above synopsis.

1. E. TRIANDRA, Schkuhr. This European species, as stated above, is said by Seubert to be Chilian, and in Bertero's collection. Naudin has accordingly introduced it into Gay's Flora Chilena, and also founded *E. Chilensis* on Bertero's specimens, describing the leaves as oblong-obovate. I have no Chilian specimens. All the American

specimens which I refer to this species were collected by E. Hall: one in ponds at Athens, Illinois, the others on the Platte River, either in Nebraska or Colorado; but I have seen others from the Mississippi Valley. They answer to Seubert's "forma intermedia" in the shape of the leaves (except that they are not "remote crenulata") and in the alternate flowers. Apparently the petals have not expanded. Search should be made in drier ground for the terrestrial form; which, in Europe, has more numerous and opposite flowers with expanded reddish petals, "petala rubella."

We have no specimens of the New Zealand and Australian species, *E. gratioloides*, A. Cunn., which should belong to the above rather than to the following species.

2. E. AMERICANA, Arn. This is the proper specific name for our common species, not only because it is the *Peplis Americana* of Pursh, but because Arnott's *Elatine Americana* was published in the year 1830, Fischer and Meyer's *E. minima* in 1836. This is the only species we have on the Atlantic border, from New Hampshire to Virginia; and we have it also from Colorado and from Oregon. It is not rarely trimerous, especially the terrestrial form. Like *E. triandra* in Europe, this also, when well developed in drier soil, has larger and broader petals than in the figure by Sprague in Gray, Gen. Ill. i. t. 95, open in anthesis and remaining so, and tinged with rose-color or purple. A diminutive form of this state is *E. Clintoniana*, Peck in 22d Report of Regents of the University of the State of N. Y., 1870, p. 53; and this is the first indication in this country of the state in question. Now that Mr. Peck has well-formed seeds of both forms, he is convinced that his *E. Clintoniana* is merely a form of *E. Americana*. The only other specimens we have of the open-flowered and commonly reddish-petalled *E. Americana* are large ones collected near New Haven, Connecticut, in October, 1873, by Dr. F. W. Hall, and communicated by Professor Eaton; and a depressed state, from Multnomah Co., Oregon, no. 134 of a collection made by T. W. Howell, and recently distributed by Mr. Woolson. In good fruits, the seeds are rather more numerous than in the figure above referred to, rather longer in proportion, and commonly more decidedly curved: in insertion they are not so basal, yet all are ascending.

3. E. BRACHYSPERMA. Our specimens are mostly terrestrial; and are from Illinois, E. Hall, Texas, E. Hall, 1872, no. 37 of his Texan distributed collection, and California, no. 257 of Kellogg and Harford's distributed collection; the latter with expanded flowers. A submersed or floating form, with narrower leaves, was collected by

M. S. Bebb in Illinois. The foliage is intermediate between the two foregoing species; the seeds are quite peculiar.

4. E. CALIFORNICA. This American representative of *E. Hydropiper* has been collected only by Mr. J. G. Lemmon, in Sierra Valley, on the Sierra Nevada, alt. 5000 feet, and was received in 1877. The seeds are just those of the European species referred to; the leaves broader and shorter-petioled; the pedicels all shorter than the calyx; the upper or younger flowers nearly sessile. These are evidently expanded in anthesis, are white, and not longer than the sepals.

2. *Two New Genera of Acanthaceæ.*

CARLOWRIGHTIA, nov. gen. Tr. *Justiciearum.*

Calyx alte divisus; segmentis angustis æqualibus. Corolla limbo 4-partito rotato, tubo tenui 2–3-plo longiore, fauce haud ampliato: lobi oblongi, consimiles, patentissimi, plani, vel posticus (æstivatione intimus) primum concavus minus patens. Stamina 2, fauci inserta: filamenta filiformia, corollæ lobis æquilonga: antheræ biloculares, loculis parallelis contiguis muticis. Staminodia nulla. Stylus filiformis: stigma capitellatum vel emarginatum. Ovarium loculis biovulatis. Capsula ovata, acuminata, obcompressa, longe clavato-stipitata. Semina plana, scabrida. — Fruticuli Texano-Arizonici, ramosissimi, glabelli; ramulis gracilibus; foliis parvis angustis integerrimis, bracteis bracteolisque consimilibus; floribus sparsis; corolla roseo-purpurea.

C. LINEARIFOLIA. Pedalis, ericoideo-foliosa; foliis angustissime linearibus; floribus paniculato-racemosis; sepalis fere discretis bracteis foliisque similibus; corollæ lobis tubo duplo longioribus; filamentis puberulo-hirtellis; antheris sagittatis, loculis basi obtusissimis; stipite capsulæ æquilongo. — *Shaueria linearifolia*, Torr. Bot. Mex. Bound. 123. *Dianthera* sp. Benth. & Hook. Gen. ii. 1114. — S. W. Texas, C. Wright, Bigelow, Parry.

C. ARIZONICA. Humilis, diffusa; foliis oblongis lanceolatis (lin. 2–3 longis); floribus in ramulis filiformibus nudis sparsis sessilibus; bracteis subulatis calyce 5-fido brevioribus; bracteolis fere nullis; corollæ lobis tubo 3-plo longioribus, 3 angusto-oblongis, postico sursum latiore basi attenuato facie macula lutea; filamentis glabris; antheris oblongis; stipite capsula breviore. — Arizona, on rocks near Camp Grant, Dr. Palmer, coll. 1867.

These two little shrubs appear to form a well-marked genus, certainly different from *Dianthera* as well as from *Schaueria*. Notwith-

standing the *Wrightia* of Brown, by an allowable and fairly euphonious combination of baptismal and surname, I am able to dedicate this genus to *Mr. Charles Wright*, the first discoverer of one of the species, my esteemed associate in botanical pursuits for more than thirty years, and one of the most indefatigable of explorers. Although he has made important and well-known collections in various parts of the world, especially in Japan, and above all in Cuba, and although a large number of species bear his name as their discoverer, it is still most proper that, in the generic form, it should be connected with the flora of North America, with the region in which his botanical investigations began, and which he most assiduously and largely explored.

GATESIA, nov. gen. Tr. *Justiciearum.*

Calyx 5-partitus, subglumaceus; segmentis setaceo-subulatis, quinto minore. Corolla subhypocraterimorphis; tubo gracili, fauce parum ampliato; limbo alte 4-lobo; lobis fere similibus planis ovatis. Stamina 2, inclusa: antheræ loculi oblongi, mutici, conformes, contigui, uno demissiore obliquo. Staminodia nulla. Stigma capitellatum. Capsula et semina *Dianthera.* Spicæ breves, floribus substrobilaceo-bracteatis *Tetramerii*, bracteolis majusculis herbaceis *Dicliptera.*

G. LÆTE-VIRENS. — *Justicia læte-virens*, Buckley in Am. Jour. Sci. xlv. (1842), 176. *Rhytiglossa viridiflora*, Nees in DC. Prodr. xi. 346. "*Justicia viridiflora*," Buckley in herb. Hook., meant for *J. viridifolia. Dicliptera Halei*, Riddell, Cat. Fl. Ludov. 1852; Chapm. Fl. 305. — Northern Alabama and Southern Tennessee to Eastern Texas.

In memory of *Dr. Hezekiah Gates*, who almost fifty years ago made and distributed a considerable collection of Alabama plants, mostly from the vicinity of Mobile. Among them was *Petalostemon corymbosus*, upon which the late Professor Bertoloni of Bologna (mistaking it for a *Composita!*) founded his genus *Gatesia.* The present genus, I trust, is better founded, and will keep up the name in connection with an Alabamian plant.

3. *New Astragali.*

ASTRAGALUS DISPERMUS. *A. didymocarpo* affinis, annuus, pusillus; ramis a basi diffusis; floribus in capitulo subgloboso 6–10; calyce albo-villoso, dentibus setaceo-subulatis tubo suo campanulato æquilongis corollam cæruleam subæquantibus; vexillo alis carinaque parum longiore; legumine calycem haud superante ovato turgido fere membranaceo venis prominulis transversis subruguloso sutura dorsali intrusa quasi didymo; locellis uniovulatis monospermis. — Wickenburg, Arizona, Dr. Palmer, 1876.

ASTRAGALUS ALLOCHROUS. Inter *Inflatos* perennes corolla ut videtur læte purpurea seu violacea insignis; floribus, etc., cæterum fere *A. Douglasii* et *macrodontis;* legumine vesicario sesquipollicari; foliolis oblongis ramisque adpresso-puberulis subcinereis. — Near Wickenburg, Arizona, Dr. E. Palmer, 1876. The flowers are 4 or 5 lines long, loosely spicate; the calyx-teeth subulate from a broad base and rather shorter than the tube; the legume minutely and sparsely puberulent, and not at all mottled. — Dr. Palmer collected an allied species the following year, on the northern border of Arizona, viz.: —

ASTRAGALUS SUBCINEREUS. *A. allochroo* et *A. Wardi* (etiam perenni?) affinis, undique molliter cinereo-pubescens; caulibus e caudice perenni diffusis ultraspithamæis foliosis; stipulis liberis parvis; petiolis brevibus; foliolis 19–21 linearibus oblongisque retusis (lin. 3–4 longis); pedunculis folio brevioribus; spica oblonga confertim multiflora; floribus patentibus (lin. 3 longis); dentibus calycis angusto-subulatis tubo campanulato paullo brevioribus; corolla ut videtur viridula apice purpurea, vexillo violaceo-striato; legumine tenuiter vesicario uniloculari globoso cum acumine parvo ⅔-pollicari puberulo purpureo-picto polyspermo, sutura ventrali modice introflexa. — Mokiak Pass in the northwestern part of Arizona, near the Utah boundary, Dr. E. Palmer, 1877.

ASTRAGALUS SCAPOSUS. Subacaulis, cæspitans, pube brevissima adpressa argenteo-incana; stipulis scariosis basi petioli adnatis; foliolis 7–11 obovato-oblongis (lin. 4–5 longis) utrinque incanis; pedunculis scapiformibus gracilibus (3–8-pollicaribus) cum spica oblonga crebriuscule 6–12-flora foliis duplo longioribus; calycis tubo oblongo-campanulato dentibus subulatis 2–3-plo longioribus; corolla "rubro-purpurea nunc albescente" (lin. 5 longa) ultra calycem vix dimidio exserta; vexillo alisque æquilongis obcordato-emarginatis carina recta obtusissima paullo longioribus; leguminibus adscendentibus parvulis (lin. 5 longis sesquilineam latis) calyce fere semi-inclusis rectis subtrigono-cylindraceis (sutura dorsali sulcata, ventrali subprominula) estipitatis canescentibus septo tenui bilocellatis; locellis 8–9-spermis. — Among rocks in dry creek-bottoms, in Mokiak Pass, near the northeastern corner of Arizona, Dr. E. Palmer, 1877. Not much like any other of our species, except that the foliage resembles that of *A. Missouriensis.*

ASTRAGALUS AMPHIOXYS. *A. Shortiano*, Nutt. (*A. cyaneo*, Gray, Pl. Fendl.) et *A. Missouriensi* sat similis; foliolis (lin. 3–6 longis) sæpius oblongis acutis; legumine tenuiter coriaceo aut anguste aut obovato-oblongo obcompresso basi apiceque attenuato et compresso nunc

leviter nunc maxime arcuato. — *A. Shortianus*, var.? *minor*, Gray, Astrag. Rev. in Proc. Am. Acad. vi. 211, magna pro parte. *A. cyaneus*, Watson in Am. Naturalist, ix. 270, quoad coll. Parry, no. 46, 49. — Southern Utah and New Mexico and Northern Arizona, Thurber, Parry, Palmer, &c. Fine specimens with better fruit than before, collected in 1877 by Dr. Palmer on the borders of Utah and Arizona, have now called proper attention to this species, which as to the foliage and flowers might be wholly mistaken for *A. Missouriensis*, while through its larger and curved legume it has been confounded with *A. Shortianus*. Mr. Watson, who noted the characters upon immature fruit, took this species to be the *A. cyaneus* of Pl. Fendl., &c.; but that, as to all the original specimens is truly *A. Shortianus*, and so this must have a new name. However it be as to the foliage and flowers, these three species are well distinguished by their fruit; *A. Missouriensis* and *A. Shortianus* by the cartilaginous (at first somewhat fleshy) texture and very abrupt obtuse or rounded base of the legume, which in the former is short, elliptical, and straight; in the latter larger and longer (one to two inches long) and curved. A marked variety of it (var. *brachylobus*, recently collected by Dr. Palmer in Arizona) has a shorter pod with an obtuse apex. *A. amphioxys*, as its name denotes, has the legume acute at both ends, the base so much narrowed that it often seems to be stipitate in the calyx, the texture is much thinner, the fore-and-aft compression greater, the arcuation moderate in the shorter pods, but strong in the longer ones.

Astragalus Mokiacensis. *A. iodantho* proximus, elatior; stipulis herbaceis; calycis dentibus tubo dimidio brevioribus; legumine oblongo rectiusculo vel parum curvato turgido ($\frac{1}{2}$–$\frac{3}{4}$-pollicari) sectione transversa ovali ad suturas levissime sulcato haud carinato. — Rocky ravines, Mokiak Pass, on the borders of Utah and N. W. Arizona, Dr. Palmer, 1877. The collector notes that the corolla is "red and white;" in the dried specimens the color is deep violet.

Astragalus ursinus. Habitu præcedentis et *A. iodanthi;* caulibus magis flexuosis; floribus minoribus (parum semipollicaribus); calycis dentibus triangulatis tubo campanulato 3–4-plo brevioribus; spica oblonga densiflora; leguminibus arrectis parvulis (semipollicaribus) oblongis sursum parum attenuatis acutis leviter arcuatis coriaceis bilocellatis, sectione transversa circulari, suturis nec sulcato-intrusis nec carinato-prominulis. — Bear Valley in south-central part of Utah, Dr. Palmer, 1877.

Astragalus triquetrus. Humilis, e radice annua diffusus, pube adpressa cinereus; stipulis parvulis scariosis liberis; foliolis 7–9 ovali-

bus oblongisve (lin. 3–4 longis); pedunculis folio multo brevioribus laxe paucifloris; floribus patentibus mox deflexis parvis (lin. 2 longis); calycis dentibus subulatis tubo brevioribus; legumine membranaceo uniloculari glabello estipitato (circ. lin. 7 longo et 4 lato) circumscriptione ovato-oblongo et arcuato sed triquetro, angulis acutis ventralibus et lateralibus, dorso lato obcompresso-impresso, suturis tenuissimis haud intrusis; seminibus 6. — Southeastern borders of Nevada, at the confluence of Muddy River with the Virgen, Dr. Palmer, 1877; sparingly collected. Habit somewhat of *A. Geyeri*, and the legume equally thin-walled; but the triquetrous form (which is restored by soaking), with the broad back somewhat impressed with a re-entering angle, is peculiar.

ASTRAGALUS SABULONUM. *A. trifloro* et *Geyeri* subsimilis, cinereo-pubescens; caulibus (spithamæis) e radice annua laxe diffusis; stipulis liberis; foliolis 9–13 oblongis (semipollicaribus); pedunculis gracilibus folio sæpe brevioribus laxe 3–5-floris; floribus lin. 3 longis; calycis subvillosi dentibus promisse subulatis tubo brevi longioribus corollam purpuream seu violaceam æquantibus; vexillo striato; legumine chartaceo subinflato oblongo-ovato subincurvo estipitato villosulo (semipollicari) prorsus uniloculari, suturis nec prominulis nec intrusis. — Southeastern border of Nevada, near the confluence of Muddy River with the Rio Virgen, on sandy ridges, Dr. Palmer, 1877.

ASTRAGALUS CONFERTIFLORUS. (*A. flavus*, var. *candicans*, Gray, Proc. Am. Acad. xii. 54.) Humilis, pube minuta adpressa canescens, e caudice crasso confertim multicaulis; stipulis scariosis folium adversus connatis; foliolis 11–13 angusto-linearibus (lin. 8–12 longis lineam latis); pedunculis strictis caules foliaque superantibus (3–4-pollicaribus); spica stricta densa multiflora (2–3-pollicari); calyce adpresso-pubescente, dentibus tubo campanulato parum brevioribus; corolla "pallide lilacina," carina apice violacea; legumine (vix semipollicari) ovali-oblongo sericeo-canescente calyce $\frac{1}{3}$-incluso modice obcompresso uniloculari, sutura dorsali intus vix tumida, ventrali extus saliente crassa. — Utah, near Richfield, L. F. Ward, in fruit; near St. George, Dr. Palmer, finely in flower; and the corolla proves to be not at all yellow. Although nearly related to *A. flavus*, it must be different. The pubescence, especially of the calyx and inflorescence, is much finer and closer; the spike more strict and dense; flowers narrower; the legume rather more exserted from the calyx, and its dorsal suture less tumid within. But the fruit of the two is very much alike: in both the retuse base is connected with the receptacle by an extremely short stipe, not longer than thick.

ASTRAGALUS TETRAPTERUS. Subpedalis e radice perenni, fere glaber; caulibus rigidis tenuiter striatis foliosis; stipulis subulatis fere liberis; foliolis 15–21 angusto-linearibus (lin. 6–10 longis parum lineam latis); pedunculis folio adæquantibus; floribus 5–9 subcapitato-congestis erectis; calycis dentibus setaceo-subulatis tubo oblongo-campanulato plus dimidio brevioribus; corolla alba angusta $\frac{2}{3}$-pollicari; legumine haud stipitato uniloculari coriaceo-oblongo (pollicari et sesquipollicari) arcuato-incurvo eximie tetraquetro-alato polyspermo, suturis haud intrusis. — Southern Utah, Mrs. Thompson and Capt. Bishop, in flower (1871–73), and now (1877) found by Dr. Palmer in fruit, 25 miles north of St. George. Remarkable for the fruit, by which it may be associated with *A. pterocarpus*, Watson, which has still more developed wings on the middle of the valves; in the present species both sutures are equally alate-carinate.

ASTRAGALUS HUMISTRATUS, Gray, var. This, from Mokiak Pass, and the same as Palmer's no. 103 of the 1876 collection, is also the same as Parry's no. 53, of 1874, named *A. Sonoræ*. But it has the fresh or freshened legume obcompressed as well as arcuate-incurved and with considerable dorsal intrusion, at least below the middle. The original specimen of *A. Sonoræ* is the only one in which the character assigned to the species holds good, and its pods are immature. It seems probable that the latter species may be suppressed, and that the former may vary remarkably in the legume and also in pubescence and length of calyx-teeth.

ASTRAGALUS PROCERUS. *Scytocarpi*, subglaber; caulibus robustis 2–3-pedalibus; stipulis parvis triangulatis liberis; foliolis ad 17 ovalibus (lin. 6–18 longis); floribus in spicam oblongam confertis numerosis patentibus mox deflexis; calycis dentibus subulato-deltoideis tubo campanulato 2–3-plo brevioribus; corolla ochroleuca (lin. 7 longis); legumine crasso-coriaceo vesicario turgide ovali pollicari obtusissimo cum mucrone basi subito breviter substipiformi-contracto, suturis nec incrassatis nec sulcatis. — Near St. Thomas, S. E. Nevada, at the confluence of Muddy River with the Virgen, among underbrush, Dr. Palmer, 1877. The legumes and the flowers resemble those of *A. Pattersonianus*; but the former are much blunter, and the calyx-teeth are very much broader and shorter.

ASTRAGALUS PREUSSII, Gray in Proc. Am. Acad. vi. 222: var. LAXIFLORUS. Floribus minoribus ($\frac{3}{4}$-pollicaribus) in spica subsparsis; calycis tubo oblongo-campanulato (nec cylindraceo) dentibus vix triplo longiore; corolla in sicco violacea; legumine minus stipitato. — Beaverdam, on the Rio Virgen, northwest corner of Arizona, Dr. Palmer

(1877), in the district where the original and single specimen of *A. Preussii* was collected by Fremont. The proper stipe of the legume is very short, or even almost wanting, but the base of the pod is abruptly contracted.

ASTRAGALUS ARTIPES. *Inflati*, glaber, e caudice perenni subspithamæus; stipulis basi petioli (nunc parce pilosuli) adnatis; foliolis 11–17 oblongis seu ovalibus (lin. 4 longis); pedunculis folio brevioribus confertim paucifloris; pedicellis suberectis calyce brevioribus; dentibus calycis elongato-subulatis tubo campanulato subæquilongis; corolla albo-purpurea; legumine tenuiter vesicario ovato purpureo-picto (pollicari) prorsus uniloculari basi parum attenuata cum stipite gracili calycem superante articulato! — Mokiak Pass, northwest corner of Arizona, in ravines, Dr. E. Palmer, 1877. A single specimen collected of this well-marked species, which is most allied to *A. ampullarius* of Watson, from the same region. The distinct articulation of the pod with its stipe is peculiar, but it occurs in *A. oophorus*, Watson, and there is some indication of such a joint in *A. leucophyllus*. The flowers are said to be creamy white, tipped with light purple. In the specimen the purple tinge predominates.

ASTRAGALUS LANCEARIUS. *Homalobi*, e caudice perenni ultrapedalis; caulibus junciformibus; stipulis parvis liberis; foliis cinereo-puberulis; petiolis nunc aphyllis nunc foliolis 2–4-jugis linearibus seu filiformibus sparsis instructis; pedunculis elongatis racemoso-plurifloris; calycis dentibus tubo campanulato dimidio brevioribus; corolla (lin. 4 longis) ut videtur alba, carina apice purpurascente; leguminibus refractis lato-lanceolatis plano-compressis glaberrimis haud stipitatis (lin. 9–15 longis 3 latis), suturis nec incrassatis nec intrusis, valvis chartaceis; seminibus 5–12. — Near Beaverdam on the Rio Virgen, northwest corner of Arizona, Dr. Palmer, 1877. Flowers and legumes about the size of those of *A. filipes*, but the latter strictly sessile in the calyx and more acute.

ASTRAGALUS CUSICKII. *Inflati*, fere glaber; caulibus (ultrapedalibus) gracilibus sparsifoliatis subflexuosis; stipulis parvulis liberis; petiolis cum rhachi junciformibus; foliolis dissitis parvis (lin. 2–6 longis, majoribus angusto-linearibus, minoribus oblongis); pedunculis elongatis laxifloris; pedicellis brevibus mox recurvis; calyce late campanulato, dentibus brevissimis fere deltoideis; corolla ut videtur alba (semipollicari); leguminibus pendulis tenuiter vesicariis (ultrapollicaribus) prorsus unilocularibus obovatis acuminulatis basi in stipitem e calyce parum exsertum attenuatis. — Union Co., in the western part of Oregon, Wm. C. Cusick, comm. by G. O. Woolson. Apparently a tall species (base of stem not seen), with somewhat the habit of *A. pictus* and of several of the *Homalobi;* the legume resembling that of *A. Whitneyi*, but of neither that nor of this species have we fully grown pods.

4. *Miscellaneæ.*

BOYKINIA ROTUNDIFOLIA, Parry in litt. Bipedalis, pilis longis arachnoideis subdeciduis et brevioribus subglanduliferis viscosis pubescens; caule ad apicem subæqualiter folioso; foliis rotundatis ambitu crenato-incisis (vix lobatis) lobulis pauci-dentatis, radicalibus caulinisque cordatis haud stipulatis, summis ovalibus grosse dentatis; pedunculis plerisque axillaribus cymam sæpius biradiatam secundifloram gerentibus; calyce hirto campanulato, lobis latis tubo brevioribus petala (æstivatione quincunciali) subæquantibus; antheris oblongis; seminibus ovali-oblongis. — San Bernardino Co., California, Parry & Lemmon, coll. 1876, no. 113.

GALIUM (RELBUNIUM) MARGARICOCCUM. E radice perenni dura diffusum, herbaceum, laxe ramosum, subglabrum; caulibus inermibus; foliis quaternis summisve tantum binis (nunc 2 intermediis minoribus) angusto-oblongis vel lato-linearibus aveniis (lin. 3–6 longis), costa nuda, marginibus tenuiter aculeolato-hirtellis; pedicellis solitariis vel subternis folia ultima 1–4-na sæpius æquantibus; corolla sordida? (fere lineam lata); fructu insigniter baccato albo (lin. 2 et ultra lato). — Dry hillsides, Calaveras to Mariposa Co., California, on the walls of the Yosemite Valley, &c. In the Botany of California, i. 283, this is mixed with *G. Nuttallii*, but no station which pertains to it is cited. It was collected during the past summer by Sir J. D. Hooker and myself, in full fruit; when the very juicy berries are pearly white and conspicuous. The color of the fruit of *G. Nuttallii* is unknown. But that is a more upright or climbing species, with broader and shorter or smaller leaves, which have conspicuously aculeolate margins, as have the angles of the stem more or less. And it belongs to the coast-region, from the Bay of San Francisco southward.

These, along with the allied North American species, belong to the section *Relbunium* of Torr. & Gray, Fl. N. Am. ii. 21, characterized by the baccate fruit from a tetramerous flower, but not to the genus *Relbunium* of Bentham and Hooker, if the character depends on an involucre unlike the foliage, and subtending a sessile or subsessile flower, as in a few S. American species. *G. microphyllum*, Gray, has the flower thus involucrate, but the involucre is similar to the proper foliage.

Aster (Machæranthera) Pattersoni. E radice bienni seu vix perenni multicaulis, subspithamæus; foliis spathulatis integerrimis (radicalibus nunc apicem rotundatum versus leviter pauci-dentatis) læte viridibus piloso-ciliatis demum glabratis; caulibus superne glanduloso-pubescentibus 2–4-foliatis 1–4-cephalis; involucri hemisphærici squamis 3–4-serialibus fere æquilongis lanceolatis acuminatis ultra medium herbaceis et squarroso-patentibus pube viscosa sæpius hirtella crebre indutis; ligulis numerosis semipollicaribus læte cæruleo-violaceis; acheniis ut videtur linearibus glabriusculis. — Colorado Rocky Mountains, at about 11,000 feet and higher along the branch of Clear Creek flowing from Torrey and Gray's Peaks, Dr. Parry, 1872; H. N. Patterson (in honor of whom the species is named), 1876; J. D. Hooker and A. Gray, 1877. A handsome and large-flowered dwarf species, with heads larger than those of *A. Coloradoensis*, and much larger than those of the related *A. Kingii*, of D. C. Eaton (which has been collected in a larger form farther south in the Wahsatch by Dr. Parry): including the rays, it has a diameter fully an inch and a half. The stems in Mr. Patterson's specimens are fully a span high. — Var.? **Hallii**, gracilior, magis ramosus; foliis ramealibus linearibus; capitulis minoribus; involucri minus pubentis squamis fere subulatis. This is mixed with *A.* (*Machæranthera*) *tanacetifolius* in the distribution of Hall and Harbour's Colorado Collection, no. 285, and indeed the heads of this form closely resemble those of that species. We found a little of it on La Veta or Sangre de Cristo Pass, S. Colorado.

Erigeron miser. Multiceps e caudice lignescente, subspithamæum, crebre villoso-pubescens, cinereum; caulibus ad apicem usque foliosis mono-oligocephalis; foliis subspathulatis integerrimis ($\frac{1}{4}$–$\frac{1}{2}$-pollicaribus); pedunculis brevibus; capitulis parvis (lin. 3 longis); involucri squamis subulatis inæqualibus; ligulis nullis; acheniis hirsutis (bicostatis); pappo exteriore setuloso sat manifesto. — Sierra Nevada, California, in crevices of rocks at Donner Lake, E. L. Greene, October, 1874; and above, on or near the summit of Mt. Stanford, J. G. Lemmon, A. Gray, and J. D. Hooker, September, 1877.

Laphamia Palmeri. *Monothrix*, sed facie capitulisque homogamis *L. rupestri* similior, ultraspithamæa e basi lignosa crassa ramosa, cinereo-pubescens; foliis submembranaceis deltoideo-subcordatis vel rotundatis grosse paucidentatis incisisve petiolatis venosis; capitulis brevipedunculatis subcymosis multifloris; involucri squamis linearibus; seta unica pappi achenio scabro hirtello æquilonga corolla paullo breviore. — At Beaverdam, in the northwest corner of Arizona, growing out of crevices of rock in cañons, in pendulous clumps, Dr. E. Palmer.

"Flowers creamy white," according to the discoverer's notes, but sulphur-yellow in the dried specimens.

THELESPERMA SUBNUDUM, Gray, Proc. Am. Acad. x. 72. Specimens collected in 1877, by Dr. Palmer, show that it commonly has ray-flowers, even of a very large size, that the akenes sometimes become granular-tuberculate, and that the pappus may develope in the manner of *T. subsimplicifolium.* The species appears to hold distinct, but it must have a new character and include *T. simplicifolium*, var. *scaposum.*

ACTINELLA BRANDEGEI, T. C. Porter. Ex affini *A. grandiflora* insigniter differt tomento minus lanato parciore; foliis simpliciter 3–5-lobatis paucisve integris glabratis; capitulis multo minoribus ebracteatis; involucri squamis lato-lanceolatis; ligulis 12–16 (tantum semipollicaribus); acheniis subturbinatis; pappi paleis firmioribus ovato-lanceolatis parum acuminatis corolla disci dimidio brevioribus. — *A. grandiflora*, var. *glabrata*, T. C. Porter, Fl. Colorado, 76. — Sangre de Cristo range of mountains and on Sierra Blanca, S. Colorado, at 11,500 feet, &c., in the alpine region, Parry (1867, undeveloped), Brandegee, Gray and Hooker. It was only in deference to my erroneous opinion that this species was omitted from publication, under the above name in 1874, in Porter and Coulter's Flora of Colorado. The species is abundantly different from *A. grandiflora*, and wholly replaces it in the alpine districts of the southern part of Colorado.

ACTINELLA BIENNIS. *A. Richardsonii* proxima, multo major; radice bienni; caule 1–2½-pedali; pedunculis monocephalis subpaniculatis; ligulis in maximis pollicem longis; disco maturo semipollicem alto; receptaculo hemisphærico et pappo *A. Richardsonii.* — S. Utah and Arizona, Mokiak Pass south of St. George, Palmer (no. 260 of coll. 1877); Richfield, Utah, L. F. Ward in Powell's Exped. no. 175, &c. Probably *A. Richardsonii*, var. *canescens*, Eaton in Watson, Bot. King; but it is uncertain whether that has not the multicipital truly perennial stock of *A. Richardsonii. A. chrysanthemoides* and *A. odorata* (the latter found along our southern frontiers) are annual species, with similar foliage, and *A. anthemoides*, the original *Hymenoxys* of Cassini (if rightly identified by Hooker and Arnott, as is wholly probable) is a rayless Bonarian species much like *A. odorata* and with a similar acutely conical receptacle.*

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* *Hymenoxys* of Cassini, and also of Bentham and Hooker, would therefore merge in *Actinella.* The only character in the diagnosis to separate them is that the receptaculum is said to be sometimes flat or convex in the former. But although Kunth and Cassini describe *H. chrysanthemoides* as having "*recep-*

ARNICA VISCOSA. Subpedalis, undique viscoso-pubescens; caulibus ad apicem usque æqualiter foliosis oligocephalis; foliis parvis (majoribus parum uncialibus) ovato-oblongis integerrimis arcte sessilibus; pedunculis brevibus; involucro disco subdimidio breviore; ligulis nullis; corollis disci ochroleucis; acheniis glabriusculis. — On Mount Shasta at about 8,000 feet, Gray & Hooker. Heads rather few-flowered, only two-thirds of an inch long. A most distinct species, seemingly not before collected, growing near the upper border of the wooded belt.

ERITRICHIUM HOLOPTERUM, Gray, Proc. Am. Acad. xii. 81, var. SUBMOLLE. Minus; spicis brevibus densifloris nudis pedunculatis cum ramis floridis parum hispidis; calyce (haud ultra lineam longo) sericeo-pubescente imberbi, lobis oblongis obtusis; nuculis angustissime alato-marginatis. — St. George, S. Utah, Dr. Palmer. The species was characterized from specimens not yet well-developed, although bearing a little nearly mature fruit. The present specimens appear to belong to it, but are smaller and lower, with better developed and ebracteate inflorescence, and calyx almost or quite destitute of setose bristles. The lower part of the plant has the same short bristles of the species, with broad papilliform base.

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taculum subplanum," it is hemispherical in age, and it is high-conical (as remarked above) in the more typical species.

As to the original *Actinella* of Persoon (*Actinea heterophylla*, Juss.), which seems out of place among the globular- and discoid-headed *Cephalophoræ*, it is intermediate in character between *Helenium* and *Actinella* of Nuttall, differing from the latter only in the looser, thinner, and smaller scales of the involucre. If, notwithstanding this, it were referred to *Actinella*, this genus would be restored to Persoon, or, under the form of *Actinea*, to Jussieu. But probably Persoon's plant should be referred to *Helenium*.

XXVI.

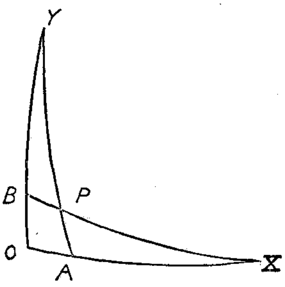
SPHERICAL CONICS.

THE THESIS OF A CANDIDATE FOR MATHEMATICAL HONORS CONFERRED WITH THE DEGREE OF A.B., AT HARVARD COLLEGE, AT COMMENCEMENT, 1877.

By Gerrit Smith Sykes.

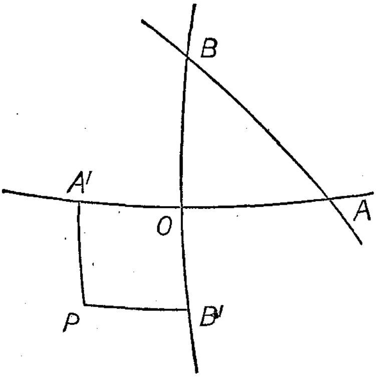
Presented by Professor Benjamin Peirce, Jan. 9, 1878.

1. It is convenient in dealing with spherical curves to have a system of spherical co-ordinates similar to plane co-ordinates. Such a system can be constructed as follows: Through the origin of plane co-ordinates draw a sphere tangent to the plane with a radius equal to unity, and project the plane axes upon the sphere by drawing lines from each point to the centre. The plane axes will thus be projected into semicircles having their extremities upon the circle of which the origin is the pole. (By circles and arcs I shall always mean great circles and their arcs, unless it is otherwise specified.) Every point on the plane will be represented by a point on the hemisphere, and this latter point can be referred to the projections of the plane axes as spherical axes. The plane co-ordinates of a point, measured on the axes, will be projected into arcs of the spherical axes, whose tangents are equal to the plane co-ordinates. The tangents of the arcs are therefore taken as spherical co-ordinates instead of the arcs themselves. Moreover, since all lines parallel to the plane axes meet them at infinity, such lines will be projected into arcs passing through the extremities of the spherical axes. Therefore, to find the spherical co-ordinates of a point on the sphere, draw arcs through the point and the extremities of the spherical axes, and take the tangents of the intercepts of these arcs as co-ordinates. Thus the co-ordinates of P are tan OA $= x$, tan OB $= y$. The spherical axes may be inclined at any angle, but I shall confine myself to rectangular axes.



2. It is evident that, in this system of co-ordinates, a straight line in the plane will be projected into an arc whose equation will be of the first degree in spherical co-ordinates, and that in general a locus of the nth degree in the plane will be represented by one of the nth degree in spherical co-ordinates. The equation of a great circle may then be written

$$Ax + By = 1. \qquad (a)$$



Let P be the pole of this circle; and let the co-ordinates of P be tan OA′ and tan OB′. But OA′ = OA — 90°, OB′ = OB — 90°. Hence, the co-ordinates of the pole are $-A$ and $-B$; that is, when the equation of a circle is written in the form (a), the co-ordinates of its pole are the negatives of the coefficients of x and y. This principle is of great utility in finding distances and equations on the sphere. The equation of a circle can also be written $y = mx + n$; and from this it can be deduced, as in plane co-ordinates, that the equation of an arc through a point $x'y'$ is

$$y - y' = m(x - x'),$$

and of that through two points is

$$\frac{y - y'}{x - x'} = \frac{y'' - y'}{x'' - x'}.$$

3. To find the distance between two points $x'y'$ and $x''y''$ on the sphere. The formulas for this, being well known, may be simply written out for reference. If ϱ denotes the distance, they are

$$\cos \varrho = \frac{1 + x'x'' + y'y''}{\pm [(1 + x'^2 + y'^2)(1 + x''^2 + y''^2)]^{\frac{1}{2}}},$$

$$\sin \varrho = \pm \left[\frac{(x'' - x')^2 + (y'' - y')^2 + (x'y'' - x''y')^2}{(1 + x'^2 + y'^2)(1 + x''^2 + y''^2)}\right]^{\frac{1}{2}},$$

$$\tan \varrho = \frac{\pm [(x'' - x')^2 + (y'' - y')^2 + (x'y'' - x''y')^2]^{\frac{1}{2}}}{1 + x'x'' + y'y''}.$$

4. A spherical conic is the intersection of a unit-sphere with a cone of the second degree, whose vertex is at the centre of the sphere. The arcs in which the cyclic planes of the cone cut the sphere are called

the cyclic arcs of the conic. Since the cone is double, it will cut the sphere in two closed curves; and we therefore name the conic differently according to the hemisphere considered. If the sphere be divided by the principal plane of the cone, it gives a closed curve whose centre will be the pole of the dividing circle, and whose principal diameters will be the arcs of the greatest and least sections of the cone. The cyclic arcs will intersect at the points where the arc of greatest section meets the dividing circle, and will be symmetrical with reference to the curve. This form of conic is a *Spherical Ellipse.*

If the sphere be divided by the plane of least section of the cone, the conic will consist of two branches. Its centre will be the pole of the dividing circle, and its principal diameters will be the arcs made by the plane of greatest section of the cone and the principal plane. The cyclic arcs meet only once, and that at the centre. This curve is the *Spherical Hyperbola*, and it will be found that its cyclic arcs have properties analogous to those of the asymptotes to the plane hyperbola.

If, again, the sphere be bisected by a plane perpendicular to the two already mentioned, there is still a third form of spherical conic, having its centre at the pole of the bisecting circle. There is, properly speaking, as might be expected from the method of projection used, no spherical parabola. If a plane parabola be projected upon a sphere, points at infinity are projected, and the spherical parabola is merely an ellipse or an hyperbola. The conic of which the major axis is a quadrant has, however, the closest analogy to the Parabola.

5. A spherical conic may also be defined as the locus of an equation of the second degree in spherical co-ordinates. The general equation is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

This can be transformed to the centre as origin; and, if we choose the principal diameters as axes, it can be reduced to the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The equation for determining the centre is a cubic, and this shows that a spherical conic has three centres. We are thus led in another way to the results arrived at in Art. 4.

This method of reducing the general equation is, however, on account of the complex formulas for transformation of spherical co-ordinates, long and tedious. It is better, therefore, to derive the equation referred to the centre from the central equation of plane conics.

By the principles explained in Art. 1, the central equation may be written $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are the tangents of the principal semi-diameters.

6. Certain properties of the spherical conics follow immediately from the quaternion equation of the cone, and it may be well to introduce the equation here. The general form of the equation, as given by Tait, is

$$S\rho\varphi\rho = 0.$$

A particular form of this is

$$\rho^2 - S\alpha\rho S\beta\rho = 0,$$

where α and β are perpendicular to the cyclic planes.

7. *To find the equations of tangent and polar arcs.*

The equation of a tangent to a spherical conic is found, as for the plane curve, by determining the value of $\frac{y'' - y'}{x'' - x'}$, when $x'' = x'$ and $y'' = y'$, and substituting and reducing. The equation is $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$. Since this represents a tangent when $x'y'$ is on the curve, it must, from the symmetry of the equation, represent the arc on which lie the points of contact of tangents from $x'y'$, if $x'y'$ is not on the curve; that is, it is the polar of $x'y'$. (When $x'y'$ is a pole with respect to the conic, I shall call it a *conic* pole, to distinguish it from the ordinary pole of circles.)

The symmetry of the equation shows that if $x''y''$ lies on the polar of $x'y'$, then $x'y'$ lies on the polar of $x''y''$. There are many properties of polars to spherical conics similar to those of plane geometry.

8. We can now find the equation of the locus of the extremity of a quadrant moving at right angles to the given conic; that is, the locus of the pole of the tangent. Thus

$$x = -\frac{x'}{a^2},\ y = -\frac{y'}{b^2},\ \text{or } x' = -a^2x,\ y' = -b^2y,$$

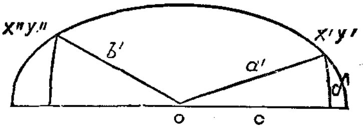
$$\text{but } \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1;$$

$$\therefore\ a^2x^2 + b^2y^2 = 1$$

is the required equation. This is a conic the tangents of whose semi-axes are $\frac{1}{a}$ and $\frac{1}{b}$: its semi-axes are therefore the complements of those of the given conic. This conic is called the *supplementary conic* of the given one. It can be combined with the given conic so as to simplify the solution of many questions in spherical conics.

9. *Conjugate Diameters.* These are related to each other as in plane conics; that is, the diameter conjugate to the one through $x'y'$ contains the conic pole of the one through $x'y'$, and *vice versa.* Its equation is therefore $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 0$. Its extremity $x''y''$ is found, as in plane conics, to be such that

$$\frac{x''}{a} = \pm \frac{y'}{b}, \frac{y''}{b} = \mp \frac{x'}{a}.$$



To find the lengths of a' and b', any two conjugate semi-diameters.

$$\cos a' = \cos c \cos \delta = \frac{1}{(1 + x'^2 + y'^2)^{\frac{1}{2}}};$$

$$\tan^2 a' = x'^2 + y'^2 = b^2 + \frac{a^2 - b^2}{a^2} x'^2;$$

and $$\tan^2 b' = x''^2 + y''^2 = a^2 - \frac{a^2 - b^2}{a^2} x'^2;$$

$$\therefore \quad \tan^2 a' + \tan^2 b' = a^2 + b^2 = \text{constant}.$$

This might also be inferred from the corresponding properties of plane conics, by the principles laid down in Art. 1.

10. To find the perpendicular distance from the centre on a tangent.

The trigonometric tangent of the perpendicular from the centre is the cotangent of the distance of the pole from the centre. Calling the perpendicular from the centre p, we have

$$\cot p = \sqrt{\frac{x'^2}{a^4} + \frac{y'^2}{b^4}} = \frac{\sqrt{\left(\frac{b^2 x'^2}{a^2} + \frac{a^2 y'^2}{b^2}\right)}}{ab} = \frac{\tan b'}{ab},$$

$$\therefore \quad \tan p = \frac{ab}{\tan b'}.$$

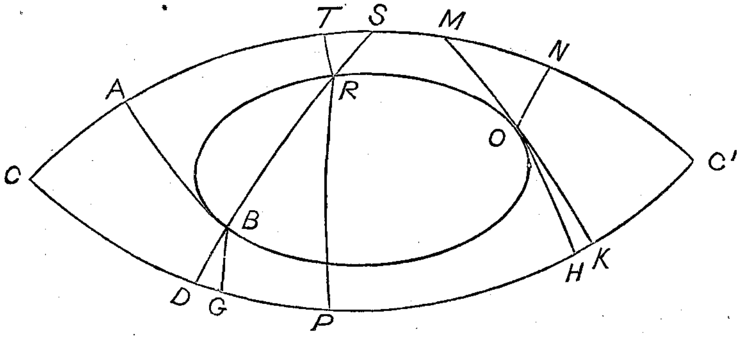
11. There are some curious properties of conics with reference to the cyclic arcs. (α) We have from the quaternion equation of the cone

$$\cos \theta \cos \theta' = k,$$

or, since α and β are perpendicular to the cyclic planes,

$$\sin \varrho \sin \varrho' = k,$$

where ϱ and ϱ' denote arcs from any point of the conic perpendicular to the cyclic arcs.



(β) If a great circle cut a spherical conic, the parts of it between the points of intersection and the respective cyclic arcs are equal. For

$$\sin \mathrm{D} = \frac{\sin \mathrm{BG}}{\sin \mathrm{BD}} = \frac{\sin \mathrm{RP}}{\sin \mathrm{RD}}, \quad \sin \mathrm{S} = \frac{\sin \mathrm{RT}}{\sin \mathrm{RS}} = \frac{\sin \mathrm{AB}}{\sin \mathrm{BS}};$$

and therefore, by (α),

$$\sin \mathrm{BS} \sin \mathrm{DB} = \sin \mathrm{RS} \sin \mathrm{DR},$$

or $$\sin (\mathrm{BR} + \mathrm{RS}) \sin \mathrm{DB} = \sin \mathrm{RS} \sin (\mathrm{DB} + \mathrm{BR});$$

$$\therefore \quad \sin \mathrm{DB} \cos \mathrm{RS} = \sin \mathrm{RS} \cos \mathrm{DB},$$

$$\therefore \quad \mathrm{DB} = \mathrm{RS}.$$

I shall also insert a quaternion proof, as given by Tait. If a conic be cut by a plane whose equation is $S\gamma\varrho = 0$, the intersections of this with the cyclic planes are $V\alpha\gamma$ and $V\beta\gamma$. Then, since a point of the curve can be reached by moving in the directions of these intersections, we may write

$$\varrho = xUV\alpha\gamma + yUV\beta\gamma,$$

$$S\alpha\varrho = yS\alpha UV\beta\gamma,$$

$$S\beta\varrho = xS\beta UV\alpha\gamma.$$

$\therefore$ $\varrho^2 - S\alpha\varrho S\beta\varrho = 0$ may be written

$$x^2 + y^2 + Axy = 0,$$

where A is a scalar function of α, β, and γ only. The form of the equation shows that any two values of x and y can be interchanged. This, then, establishes the theorem.

If the cutting arc becomes a tangent, the parts intercepted between the point of contact and the cyclic arcs are equal.

12. From the two properties announced, we can deduce another. The area of the spherical triangle formed by a tangent and the cyclic arcs is constant.

$$\sin \mathrm{ON} = \sin \mathrm{OM} \sin \mathrm{M},$$

$$\sin \mathrm{OH} = \sin \mathrm{OK} (\sin \mathrm{OKH} = \sin \mathrm{OKC}').$$

$$\therefore \quad \sin^2 \mathrm{OK} \sin \mathrm{M} \sin \mathrm{K} = \text{constant}.$$

But $$\cos \mathrm{C}' = -\cos(\mathrm{M} + \mathrm{K}) - 2 \sin^2 \mathrm{OK} \sin \mathrm{M} \sin \mathrm{K}.$$

Hence, since C' is constant for a given conic, $\mathrm{M} + \mathrm{K}$ is constant, and therefore the area of the triangle is constant, for it equals $\mathrm{C}' + \mathrm{M} + \mathrm{K} - 180°$.

We may then define a spherical conic as the envelope of the base of a spherical triangle, of which the vertex, the vertical angle, and the area are given. The arcs forming the vertical angle are the cyclic arcs of the conic.

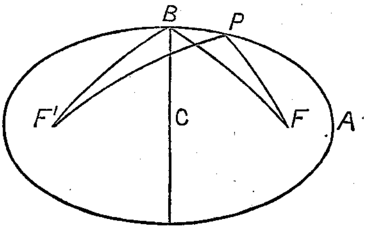
13. This property is also true of the supplementary conic; and therefore, remembering the relation existing between arcs and their poles, we may define a spherical conic as the locus of the vertex of a spherical triangle, of which the base is given in length and position, and of which the sum of the sides is also given. The extremities of the base are the *foci* of the conic; and we now wish to determine their position. It is evident that they are the poles of the supplementary cyclic arcs. Since, by Art. 11 (β), a conic is symmetrical with respect to its cyclic arcs, their poles must lie on an axis at equal distances from the centre; and, since the axes of conics and those of supplementary conics are parts of the same circles, the foci of a conic must lie on an axis of that conic, at equal distances from the centre. It can be shown that this axis is the major axis. Let F and F′ be the foci; then, as the sum of the sides of a spherical triangle is greater than the base, the foci fall inside the conic. When P is at A,

$$\mathrm{F'A} + \mathrm{FA} = \text{constant} = 2\,\mathrm{CA};$$

and, when at B,

$$\mathrm{F'B} + \mathrm{FB} = 2\,\mathrm{FB} = 2\mathrm{CA};$$

$$\therefore \quad \mathrm{CA} = \mathrm{FB}:$$



$$\cos \mathrm{FB} = \cos \mathrm{BC} \cos \mathrm{CF} = \cos \mathrm{CA},$$

$$\therefore \quad \cos \mathrm{CA} < \cos \mathrm{BC}, \qquad\qquad \therefore \quad \mathrm{CA} > \mathrm{BC}.$$

14. The equation of the conic may be determined by finding the locus of the vertex of a triangle, when the base and the sum of the sides are given.

Suppose $A + B = 2\alpha$; then

$$\cos A \cos B - \sin A \sin B = \cos 2\alpha,$$

$$(\cos A \cos B - \cos 2\alpha)^2 = (1 - \cos^2 A)(1 - \cos^2 B),$$

$$\cos^2 A + \cos^2 B - 2 \cos 2\alpha \cos A \cos B = \sin^2 2\alpha;$$

but $\cos 2\alpha = 1 - 2 \sin^2 \alpha$, $\sin 2\alpha = 2 \sin \alpha \cos \alpha$;

$$\therefore \quad (\cos A - \cos B)^2 + 4 \sin^2 \alpha \cos A \cos B = 4 \sin^2 \alpha \cos^2 \alpha.$$

Let $\mathrm{FC} = c$, and use the formulas of Art. 3; then

$$x^2 \tan^2 c + (1 - x^2 \tan^2 c) \sin^2 \alpha = (1 + x^2 + y^2) \sin^2 \alpha \cos^2 \alpha \sec^2 c.$$

But $\cos c = \dfrac{\cos \alpha}{\cos \beta}$, $\sec^2 c = \dfrac{\cos^2 \beta}{\cos^2 \alpha}$, $\tan^2 c = \dfrac{\cos^2 \beta - \cos^2 \alpha}{\cos^2 \alpha}$;

and hence, by easy reductions,

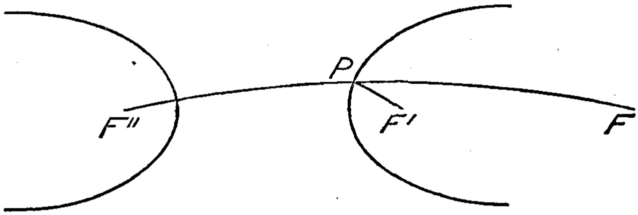
$$x^2 \cos^2 \alpha \sin^2 \beta + y^2 \sin^2 \alpha \cos^2 \beta = \sin^2 \alpha \sin^2 \beta,$$

or

$$x^2 \cot^2 \alpha + y^2 \cot^2 \beta = 1,$$

which is the same form of equation as that given in Art. 5.

15. If the sphere be so divided as to make a spherical hyperbola, then the locus becomes the vertex of a triangle whose base and also the difference of its sides are given.



$$\mathrm{F''P} + \mathrm{PF} = 180°, \quad \mathrm{F'P} + \mathrm{PF} = \text{constant};$$

$$\therefore \quad \mathrm{F''P} - \mathrm{F'P} = \text{constant}.$$

As the spherical hyperbola is simply made up of the halves of two equal ellipses, it is not necessary to deal with it separately; for whatever is true of the ellipse is true of the hyperbola.

16. There are certain relations between α, β, and c, which enable us to reduce some forms of equations.

$$\cos c = \frac{\cos\alpha}{\cos\beta}, \quad \tan\alpha = a, \quad \tan\beta = b;$$

$$\therefore \quad \frac{\sin^2 c}{\sin^2\alpha} = \frac{\dfrac{1}{b^2+1} - \dfrac{1}{a^2+1}}{\dfrac{a^2}{(b^2+1)(a^2+1)}} = \frac{a^2-b^2}{a^2} = \varepsilon_1^{\,2},$$

$$\frac{\tan^2 c}{\tan^2\alpha} = \frac{\dfrac{1}{b^2+1} - \dfrac{1}{a^2+1}}{\dfrac{a^2}{a^2+1}} = \frac{a^2-b^2}{a^2(1+b^2)} = \varepsilon^2,$$

$$\frac{\sin^2 2c}{\sin^2 2\alpha} = \frac{(a^2+1)(b^2+1) - (b^2+1)^2}{a^2} = \frac{(a^2-b^2)(1+b^2)}{a^2} = \varepsilon'^2.$$

17. By Art. 11, $\sin\varrho \sin\varrho' = k$; and from this can be proved a property of the foci similar to one in plane conics. Since the foci of a conic are the poles of the cyclic arcs of the supplementary conic, the distances of the foci from any tangent are equal to the distances of the corresponding point of the supplementary conic from its cyclic arcs. Hence, if δ and δ_1 denote the distances of the foci from a tangent, $\sin\delta \sin\delta_1 =$ constant.

This constant can be determined as follows:—Using the pole of the tangent as in Art. 10, we have

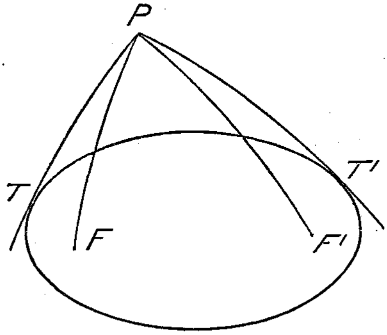
$$\sin\delta = \frac{ab^2(a-\varepsilon x')}{[(a^4b^4 + b^4x'^2 + a^4y'^2)(1+a^2\varepsilon^2)]^{\frac{1}{2}}};$$

$$a^4b^4 + b^4x'^2 + a^4y'^2 = a^2b^2\left(a^2b^2 - \frac{a^2-b^2}{a^2}\,x'^2 + a^2\right),$$

$$\frac{a^2-b^2}{a^2} = \frac{\varepsilon^2(a^2+1)}{a\varepsilon^2+1}, \quad a^2(b^2+1) = \frac{a^2(a^2+1)}{a^2\varepsilon^2+1};$$

$$\therefore \quad \sin\delta = \frac{b(a-\varepsilon x')}{[(a^2-\varepsilon^2x'^2)(a^2+1)]^{\frac{1}{2}}}, \quad \sin\delta_1 = \frac{b(a+\varepsilon x')}{[(a^2-\varepsilon^2x'^2)(a^2+1)]^{\frac{1}{2}}},$$

$$\sin\delta \sin\delta_1 = \frac{b^2}{a^2+1}.$$



18. The two tangents drawn from any point to a spherical conic make equal angles with the arcs joining that point to the foci. The poles of PT and PT' lie on the supplementary conic; the poles of PF and PF' lie on the supplementary cyclic arcs. Moreover, all these poles lie on the circle of which P is the pole. Hence, by Art. 11 (β), the poles of PT and PF are the same distance apart as those of PT' and PF', and therefore the angles TPF and T'PF' are equal.

When the point P is on the conic, this theorem becomes the following, which was one of the first discovered properties of spherical conics: — The two arcs from the foci to any point of the conic make equal angles with the tangent at that point. It follows immediately also that, if from any point of one of two confocal conics tangent arcs be drawn to the other, these tangents make equal angles with the tangent to the first conic at the given point.

19. The theorem of Art. 18 also proves that, if two confocal conics intersect, they intersect at right angles.

Since two confocal conics imply two supplementary concyclic conics, we see that, if a common tangent arc be drawn to two concyclic conics, the part intercepted between the two points of contact will be a quadrant. This is evident from the first part of this article, for this arc measures the right angle made by the two confocal conics.

20. We now come to an important theorem: — The projection of a spherical conic on a tangent plane to the sphere at one of the foci is a plane conic having the point of contact for a focus. Let $\varrho + \varrho' = 2\alpha$; then if $\theta =$ the angle which ϱ makes with the axis,

$$\cos \varrho' = \cos 2\alpha \cos \varrho + \sin 2\alpha \sin \varrho\,;$$

but we also have by spherical trigonometry

$$\cos \varrho' = \cos \varrho \cos 2c + \sin \varrho \sin 2c \cos \theta,$$

$$\therefore \quad \tan \varrho = \frac{\cos 2c - \cos 2a}{\sin 2a - \sin 2c \cos \theta},$$

which is the polar equation of a spherical conic referred to a focus as pole. Since $\tan \varrho$ is the projection of ϱ on the plane, and θ remains the same in the plane, — being the angle of the tangents to the sphere at the focus, — the equation shows that the projection on the plane is a conic with the same focus and an eccentricity equal to $\frac{\sin 2c}{\sin 2a}$.

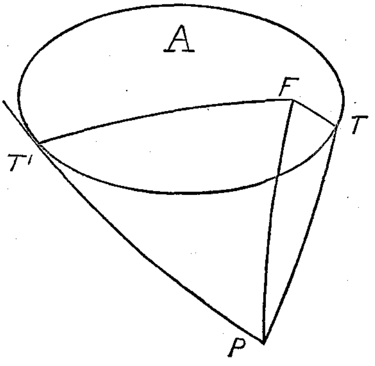
The importance of this principle is due to the fact that it enables us to establish many properties of spherical conics with reference to a single focus from known properties of plane conics.

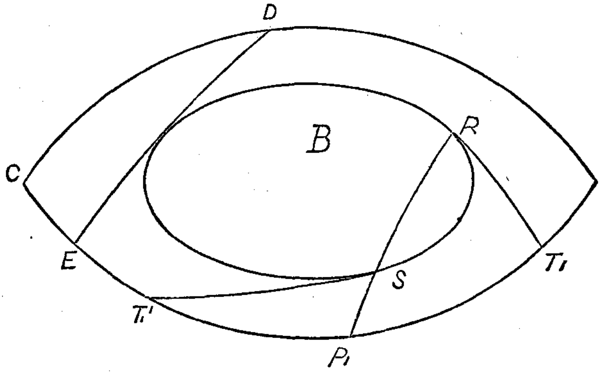
21. The angle subtended at the focus by any chord is bisected by the arc joining the focus to its conic pole. The spherical conic can be projected into a plane conic having the same focus, of which this property is true; but the angles of the plane conic at the focus are the measures of the angles of the spherical conic.

From this can be established a reciprocal property by the aid of the supplementary conic.

Two tangent arcs to a conic and the arc joining their points of contact cut the cyclic arc in three points, one of which bisects the distances between the other two.

Suppose A represents a conic and B its supplementary conic, and $\angle T'FP = \angle TFP$. The poles of FT', FP, and FT all lie on a cyclic arc of the supplementary conic. Let these poles be T'_1, P_1, and T_1 respectively; then $\widehat{T'_1P_1} = \widehat{P_1T_1}$. The poles of PT and PT' lie on the supplementary conic; denote them by R and S. Then R, S, and P_1 lie on the circle of which P is the pole. Moreover, T_1' and S lie on the circle of which





T′ is the pole, and T_1 and R on that of which T is the pole; but the circles of which T′ and T are the poles are tangents to the supplementary conic. Hence the theorem is established.

22. The directrix, or director arc, is the conic polar of the focus. We have, as a particular case of Art. 21: The arc joining the focus to the conic pole of any arc passing through the focus is perpendicular to the latter arc.

Also every tangent to a conic and the arc joining its point of contact to the conic pole of a cyclic arc meet that cyclic arc in two points a quadrant apart.

By the theory of projections, explained in Art. 21, we have: The arcs drawn from the focus of a conic to the point of intersection of two tangents, and to the point where the arc through the points of contact meets the director arc, are at right angles to each other; or, in other words, if any chord PP′ cut the directrix in D, then FD is the external bisector of $\angle$PFP′.

Then, by the same reasoning as in Art. 21, the converse of this can be proved. The arc passing through the conic pole of a cyclic arc and through the point of intersection of two tangent arcs meets the cyclic arc at a point a quadrant distant from that at which the cyclic arc is met by the arc joining the points of contact of these tangents.

23. The angle subtended at the focus by the part cut off on a variable tangent by two fixed tangents is constant. This is a right angle, if the two fixed tangents intersect on the directrix. If through two fixed points on a conic two arcs be drawn intersecting in a third point of the conic, they intercept a constant segment on the cyclic arc. This is a quadrant, if the arc joining the two fixed points passes through the conic pole of the cyclic arc.

Also, from the corresponding property of plane conics, the sum of the cotangents of the segments of a focal chord is constant. Reciprocally: — If, from a point upon the cyclic arc, tangents be drawn to a conic, the sum of the trigonometric cotangents of the angles which they make with the cyclic arc is constant; for these angles are measured by the segments of the focal chord of the supplementary conic.

The rectangle under the tangents of the segments of a focal chord is proportional to the sum of the tangents of the segments.

If, from a point upon the cyclic arc, tangents be drawn to a conic, the product of the trigonometric tangents of the angles which they make with the cyclic arc is proportional to the sum of the tangents of the angles.

Many other properties and their reciprocals can in the same way be deduced from known properties of plane conics.

24. The directrix being the conic polar of the focus, its equation is

$$\frac{x \tan c}{a^2} = 1, \quad \text{or } \frac{\varepsilon}{a}\, x = 1.$$

From this it can be proved that the sines of the distances of any point of the conic from a focus and the corresponding directrix are in a constant ratio. By the formulas of Art. 3, if ϱ denotes the distance of any point $x'y'$ of the conic from the directrix, then

$$\sin \varrho = \frac{a \mp \varepsilon x'}{[(a^2 + \varepsilon^2)(1 + x'^2 + y'^2)]^{\frac{1}{2}}}.$$

If ϱ' denotes the focal distance of $x'y'$, then

$$\sin \varrho' = \frac{[(a\varepsilon \mp x')^2 + (1 + a^2\varepsilon^2)y'^2]^{\frac{1}{2}}}{[(1 + a^2\varepsilon^2)(1 + x'^2 + y'^2)]^{\frac{1}{2}}};$$

but

$$y'^2 = \frac{b^2}{a^2}\,(a^2 - x'^2),$$

$$= \frac{1 - \varepsilon^2}{1 + a^2\varepsilon^2}\,(a^2 - x'^2),$$

$$\therefore \quad \sin \varrho' = \frac{a \mp \varepsilon x'}{[(1 + a^2\varepsilon^2)(1 + x'^2 + y'^2)]^{\frac{1}{2}}},$$

$$\therefore \quad \frac{\sin \rho'}{\sin \rho} = \sqrt{\frac{a^2 + \varepsilon^2}{1 + a^2\varepsilon^2}}.$$

Then, by the method already used several times, we have the reciprocal: — In a conic, the sine of the angle which a tangent to the curve makes with the cyclic arc is in a constant ratio to the sine of the distance of this tangent arc from the conic pole of the cyclic arc.

25. The equation of an arc can be written in the form

$$x \cos \alpha + y \sin \alpha = \tan p;$$

and from this it can, as in plane conics, be shown that the perpendicular from the centre on a tangent satisfies the equation

$$\tan^2 p = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha.$$

This property cannot, as in plane conics, be used to find the locus of the intersection of tangents at right angles to each other. The latter problem can best be solved by finding the cone which is the locus of the intersection of tangent planes to a given cone at right angles to each other.

If $S\rho\phi\rho = 0$ be the equation of a cone, $S\varpi\phi\rho = 0$ is the equation of a tangent plane. Suppose ϖ is the line of intersection, and α, β, and γ are three rectangular unit vectors; then for different sides of the cone $\rho = \varpi + x\alpha$, $\rho_1 = \varpi + y\beta$, $\rho_2 = \varpi + z\gamma$, and x, y, and z are eliminated by the fact that, when $\varpi + x\alpha$, &c., are substituted in the equation of the cone, the roots of the quadratics in x, y, and z are equal, because α, β, and γ are in the tangent planes. Substituting $\varpi + x\alpha$ for ρ in $S\rho\phi\rho = 0$, we have

$$S(\varpi + x\alpha)\phi(\varpi + x\alpha) = S\varpi\phi\varpi + 2xS\alpha\phi\varpi + x^2 S\alpha\phi\alpha = 0\,;$$

and, since the roots are equal,

$$S^2\alpha\phi\varpi = S\varpi\phi\varpi.S\alpha\phi\alpha.$$

By introducing the other values of ρ, we get also

$$S^2\beta\phi\varpi = S\varpi\phi\varpi.S\beta\phi\beta,$$

$$S^2\gamma\phi\varpi = S\varpi\phi\varpi.S\gamma\phi\gamma.$$

$$\therefore\ S^2\alpha\phi\varpi + S^2\beta\phi\varpi + S^2\gamma\phi\varpi = (S\alpha\phi\alpha + S\beta\phi\beta + S\gamma\phi\gamma)S\varpi\phi\varpi.$$

But it is known that

$$\phi\varpi = \frac{iSi\varpi}{a^2} + \frac{jSj\varpi}{b^2} + \frac{kSk\varpi}{c^2}.$$

Hence we have

$$(\phi\varpi)^2 = -\frac{S^2 i\varpi}{a^4} - \frac{S^2 j\varpi}{b^4} - \frac{S^2 k\varpi}{c^4} = S\varpi\phi^2\varpi,$$

$$S^2\alpha\phi\varpi = \frac{S^2\alpha i S^2 i\varpi}{a^4} + \frac{S^2\alpha j S^2 j\varpi}{b^4} + \frac{S^2\alpha k S^2 k\varpi}{c^4}$$
$$+ 2\left(\frac{S\alpha i S\alpha j S i\varpi S j\varpi}{a^2 b^2} + \frac{S\alpha j S\alpha k S j\varpi S k\varpi}{b^2 c^2} + \frac{S\alpha k S\alpha i S k\varpi S i\varpi}{c^2 a^2}\right),$$

with similar expressions for $S^2\beta\phi\varpi$ and $S^2\gamma\phi\varpi$. Then remembering that

$$S^2\alpha i + S^2\beta i + S^2\gamma i = -i^2 = 1,$$

$$S\alpha i S\alpha j + S\alpha j S\alpha k + S\alpha k S\alpha i = 0,$$

we have

$$S^2\alpha\phi\varpi + S^2\beta\phi\varpi + S^2\gamma\phi\varpi = -(\phi\varpi)^2 = -S\varpi\phi^2\varpi.$$

$$\text{Also } S\alpha\phi\alpha = \frac{S^2\alpha i}{a^2} + \frac{S^2\alpha j}{b^2} + \frac{S^2\alpha k}{c^2}\,;$$

$$\therefore\ S\alpha\phi\alpha + S\beta\phi\beta + S\gamma\phi\gamma = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

The equation of the cone sought becomes then

$$-S\varpi\varphi^2\varpi = \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)S\varpi\varphi\varpi;$$

but $$S\varpi\varphi\varpi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2},$$

and $$-S\varpi\varphi^2\varpi = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4};$$

$$\therefore\ \left(\frac{1}{b^2} + \frac{1}{c^2}\right)\frac{x^2}{a^2} + \left(\frac{1}{a^2} + \frac{1}{c^2}\right)\frac{y^2}{b^2} + \left(\frac{1}{a^2} + \frac{1}{b^2}\right)\frac{z^2}{c^2} = 0, \text{ or}$$

$$(b^2 + c^2)x^2 + (a^2 + c^2)y^2 + (a^2 + b^2)z^2 = 0, \qquad (1)$$

is the equation of the required cone.

Now the equation of the supplementary cone being $S\tau\varphi^{-1}\tau = 0$, where $\tau = \varphi\rho$, its equation can be written

$$a^2x^2 + b^2y^2 + c^2z^2 = 0. \qquad (2)$$

In any cone, as $Mx^2 + Ny^2 + Pz^2 = 0$, the equation of the cyclic planes containing the axis of x, for example, is

$$y^2(N - M) + z^2(P - M) = 0,$$

so that (1) and (2) have the same cyclic planes. From this it follows that the locus of the intersection of two tangents to a spherical conic which meet at right angles is a conic concyclic with the conic supplementary to the given conic. Reciprocally: — Since the poles of these tangents are on the supplementary conic a quadrant apart, it results that the locus of the pole of a chord of 90° is a conic concyclic with the given conic, and the chord envelopes a conic confocal with the conic supplementary to the given one.

26. The locus of the intersection of tangents at the extremities of conjugate diameters can also be found. The equations of the tangents are

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

$$\frac{xx''}{a^2} + \frac{yy''}{b^2} = 1, \text{ or, by Art. 9, } \frac{xy'}{ab} - \frac{x'y}{ab} = 1;$$

then, squaring and adding,

$$\frac{x^2}{a^2}\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right) + \frac{y^2}{b^2}\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right) = 2,$$

$$\text{or } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2,$$

which is a spherical conic.

27. Several additional properties of cyclic arcs can also be stated. First, to find their equation. Their poles are at the same distance from the centre on the axis of y as the foci of the supplementary conic are from its centre. Suppose α, β, c, and α', β', c', are the semi-diameters and focal distances of the two conics, then

$$\alpha' + \beta = \frac{\text{ə}}{2}, \; \alpha + \beta' = \frac{\text{ə}}{2},$$

$$\cos c' = \frac{\cos a'}{\cos \beta'} = \frac{\sin \beta}{\sin a},$$

$$\tan^2 c' = \frac{a^2 - b^2}{b^2(a^2 + 1)}.$$

Hence, the co-ordinates of the poles of the cyclic arcs are

$$0 \text{ and } \pm \sqrt{\frac{a^2 - b^2}{b^2(a^2 + 1)}};$$

and the equation of the arcs can (Art. 2) be written

$$y = \mp b \sqrt{\frac{a^2 + 1}{a^2 - b^2}}.$$

Let $2\theta =$ the angle between the cyclic arcs; then $2\theta + 2c' = \text{ə}$. Hence,

$$\sin \theta = \cos c' = \frac{\sin \beta}{\sin a},$$

so that the angle of the cyclic arcs is equal to the angle subtended by the minor axis at either focus.

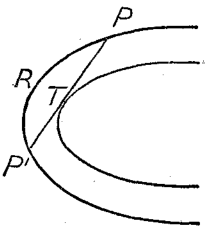
28. Every tangent to a spherical conic cuts the cyclic arcs in two points, such that the product of the trigonometric tangents of the halves of the arcs lying between these points and the point of intersection of the cyclic arcs is constant. It is known by spherical trigonometry that (see fig. B, Art. 21)

$$\tan (\text{area } \textsc{ced}) = \frac{\tan \frac{1}{2} \textsc{cd} \tan \frac{1}{2} \textsc{ce} \sin \textsc{c}}{1 + \tan \frac{1}{2} \textsc{cd} \tan \frac{1}{2} \textsc{ce} \cos \textsc{c}}.$$

$$\therefore \text{(by Art. 12) } \tan \tfrac{1}{2} \textsc{cd} \tan \tfrac{1}{2} \textsc{ce} = \text{constant.}$$

If arcs be drawn from any point of a spherical conic to the foci, the product of the tangents of the halves of the angles made by these arcs with the major axis is constant. For the intersection of the supplementary cyclic arcs is the pole of the major axis, and the poles of the arcs drawn from the foci lie on these cyclic arcs, at the extremities of a tangent to the supplementary conic. If the angles be measured in opposite directions, the ratio of the tangents of the semi-angles is constant; and a similar modification may be made in the reciprocal theorem.

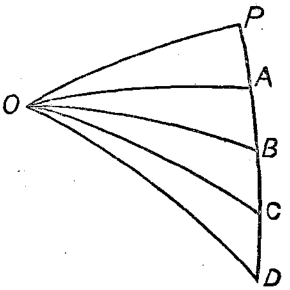
29. If a tangent be drawn to the inner of two concyclic conics, the parts included between the point of contact and the outer curve are equal. This is an immediate consequence of Arts. 11 and 12. Then, by the method of infinitesimals used in plane conics (Salmon's Conic Sections, § 396), the area included between the tangent and the outer curve is constant, as the point of contact moves along the inner curve.



The conics supplementary to concyclic conics are confocal. The pole T′ of PP′ lies upon the outer conic, and, if from T′ tangents be drawn to the inner conic, these tangents measure the angles which the tangents at P and P′ make with PP′. Moreover, the curve between the points of contact of tangents from successive positions of T′ measures the infinitely small angle made by consecutive tangents along the curve PRP′. But the sum of these angles with those at P and P′ mentioned above is constant. Hence this theorem follows: — If, from a point on the outer of two confocal conics, tangents be drawn to the inner one, the sum of these tangents and of the concave part of the curve included between them is constant.

30. I shall now give some principles of conic poles and polars with reference to spherical conics. The spherical anharmonic ratio is

$$\frac{\sin \mathrm{AD} \sin \mathrm{BC}}{\sin \mathrm{AB} \sin \mathrm{CD}}$$



To prove this ratio constant for any given pencil,

$$\sin \mathrm{AD} = \frac{\sin \mathrm{AOD} \sin \mathrm{OA}}{\sin \mathrm{ADO}} = \frac{\sin \mathrm{AOD} \sin \mathrm{OA} \sin \mathrm{OD}}{\sin \mathrm{OP}};$$

and, obtaining corresponding values for sin BC, &c., we have

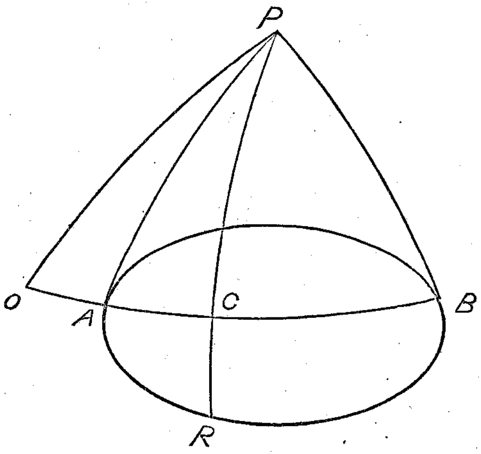
$$\frac{\sin \mathrm{AD} \sin \mathrm{BC}}{\sin \mathrm{AB} \sin \mathrm{CD}} = \frac{\sin \mathrm{AOD} \sin \mathrm{BOC}}{\sin \mathrm{AOB} \sin \mathrm{COD}};$$

so that the ratio depends only on the angles at O, and is constant for any given pencil.

31. To prove that an arc drawn from a conic pole O is harmonically divided by the point O, the conic, and the polar of O, as defined in Art. 7.

I shall deduce the proof from the corresponding property of plane conics.

Let O be a pole, and PR its polar; then, from any point of the polar as P, there will radiate a spherical pencil.



Project the spherical conic upon a tangent plane at P; and suppose O, A, C, and B to be projected into O′, A′, C′, and B′. It is evident, then, that O′ is the pole of PC′, and

$$\therefore \quad \frac{\mathrm{O'B'}.\mathrm{A'C'}}{\mathrm{O'A'}.\mathrm{C'B'}} = 1.$$

This ratio, however, depends upon the sines of O′PA′, &c.; hence, since all lines on the plane are perpendicular to the radius drawn to P,

$$\frac{\sin \mathrm{OPB} \sin \mathrm{APC}}{\sin \mathrm{OPA} \sin \mathrm{CPB}} = \frac{\sin \mathrm{OB} \sin \mathrm{AC}}{\sin \mathrm{OA} \sin \mathrm{CB}} = 1.$$

32. Let OB $= \varrho_1$, OA $= \varrho_2$, OC $= \varrho$; then

$$\frac{\sin (\rho - \rho_2)}{\sin (\rho_1 - \rho)} = \frac{\sin \rho_2}{\sin \rho_1},$$

$$\therefore \quad \frac{\tan \rho - \tan \rho_2}{\tan \rho_1 - \tan \rho} = \frac{\tan \rho_2}{\tan \rho_1},$$

$$\therefore \quad \frac{1}{\tan \rho} = \tfrac{1}{2} \left(\frac{1}{\tan \rho_1} + \frac{1}{\tan \rho_2} \right).$$

From this can be found the equation of the polar of the origin. The general equation of a spherical conic, transformed to spherical polar co-ordinates by

$$x = \cos\theta \tan\varrho,\ y = \sin\theta \tan\varrho,$$

becomes

$$(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)\tan^2\varrho$$
$$+ 2\,(g\cos\theta + f\sin\theta)\tan\varrho + c = 0.$$

Then using $\tan\varrho$ as the variable, we find the equation of the polar, by the same process as in plane conics, to be

$$gx + fy + c = 0.$$

From this equation, it is evident that the conic polar of the centre is the circle of which the centre is the spherical pole. This can also be proved by putting OA = OB in the last article. This circle corresponds to the line at infinity in plane conics.

33. The condition that three arcs meet in a point, and the equation of an arc through the intersection of two other arcs, are the same in spherical co-ordinates as for lines in plane co-ordinates. We can then prove, exactly as in plane conics, the following theorems:— Draw any two arcs through a point O; join directly and transversely the points where these arcs cut the conic. Then, if the direct arcs intersect in P, and the transverse in R, the arc PR is the polar of O.

The lines joining the corresponding vertices of a spherical triangle and its conjugate meet in a point.

If a quadrilateral be inscribed in a conic, each of the three points of intersection of the diagonals will be the conic pole of the arc joining the other two.

All of these properties are also seen to be true by projections on a plane.

34. The polar of $x'y'$ relatively to a conic is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

of which the spherical pole is $\left(-\frac{x'}{a^2} - \frac{y'}{b^2}\right)$; the polar of $x''y''$ relatively to the supplementary conic is

$$a^2xx'' + b^2yy'' = 1\,;$$

and, if this is the polar of $\left(-\frac{x'}{a^2} - \frac{y'}{b^2}\right)$, it becomes

$$xx' + yy' = -1,$$

whose spherical pole is $x'y'$. Hence, to a point and its polar with reference to a conic, there correspond an arc and its conic pole with reference to the supplementary conic.

35. From Art. 30 can be deduced: *If, from four fixed points on* a spherical conic, arcs be drawn to a fifth point of the conic, their anharmonic ratio is constant. Reciprocally: If four fixed tangents be drawn to a conic, they will cut a fifth tangent in four points whose anharmonic ratio is constant.

36. If the spherical conic be projected upon a plane tangent to the sphere at the pole of a cyclic arc, the conic becomes a plane circle, and the cyclic arc a line at infinity. The plane passing through the centre of the sphere parallel to the plane of projection is the cyclic plane; and, if two planes be drawn through the centre of the sphere and through any two lines in the plane of projection, these two planes will intersect the parallel cyclic plane in two radii making the same angle as the lines. Since the centre of the circle is the pole of the line at infinity, its projection on the sphere will be the conic pole of the cyclic arc. By means of this method, many properties of the circle can be extended, with suitable modifications, to spherical conics. The propositions of Arts. 21 and 22 can be proved in this way, though in the inverse order.

37. If two tangents to a conic intercept upon a cyclic arc a segment of constant length, the locus of the point of intersection of these tangents is a second conic, and the arc joining the points of contact of the tangents will envelope a third conic. The cyclic arc will be a cyclic arc of the new conics, and will have the same conic pole for all three conics. For, if two tangents to a circle make a constant angle, the locus of the intersection is a circle, and the chord of contact envelopes a third circle, and these three circles are all concentric. Reciprocally: If a constant angle has its vertex at either focus of a spherical conic, the arc joining the points in which the sides of the angle cut the curve will envelope a second conic, and the tangents to the given conic at the points of cutting will intersect on a third conic, then (Art. 34) the focus at which the vertex of the constant angle is placed will be a focus for the three conics, and the directrix will also be the same for all three.

This example is sufficient to illustrate the method of applying the principle. It is plain that all graphic properties of the circle can be extended to spherical conics.

38. If now we suppose the radius of the sphere to become infinite, the spherical conics become plane. As the properties already proved still hold good, we can deduce the well-known properties of plane conics from corresponding ones of spherical conics. A remarkable analogy has been shown to exist between the foci and cyclic arcs; and, as the properties of the foci hold good in the plane conics, the question naturally arises, What becomes of the reciprocal properties of the cyclic arcs? In the case of the ellipse, the cyclic arcs become lines at infinity; but, in the hyperbola, the cyclic arcs become the plane asymptotes.

The Calculus can be applied to the equations of the spherical conics, and expressions can thus be found for the area and arc. As these, however, involve elliptic integrals, I have omitted them.

XXVII.

ON THE INFLUENCE OF INTERNAL FRICTION UPON THE CORRECTION OF THE LENGTH OF THE SECONDS' PENDULUM FOR THE FLEXIBILITY OF THE SUPPORT.

By C. S. Peirce.

[Communicated by the authority of the Superintendent of the Coast Survey.]

It has been shown by Professor A. M. Mayer that the only sensible resistance to the motion of a tuning-fork is proportional to the velocity. In the case of a slowly vibrating body, the chief effect is probably due to that lagging of the strain after the stress, which Weber has called the elastic after-effect (*Nachwirkung*). The influence of the former mode of resistance upon the period of oscillation of a pendulum oscillating on an elastic tripod is easily calculated. The same thing cannot, in my opinion, be effected for the other kind of resistance, in the present state of our knowledge; nevertheless, the main characteristics of the motion may be made out. Put

t, for the time;

φ, for the instantaneous angle of deflection of the pendulum;

s, for the instantaneous horizontal displacement of the knife-edge from its position of equilibrium, in consequence of the flexure of the support;

l, for the length of the corresponding simple pendulum;

h, for the distance from the knife-edge to the centre of mass of the pendulum;

g, for the acceleration of gravity;

γ, for the ratio of g to the statical displacement of the point of support, which would be produced by a horizontal force equal to the weight of the pendulum;

a, for the coefficient of internal friction supposed proportional to the velocity.

Then the differential equations are

$$lD^2_t\varphi + D^2_ts = -g\varphi$$
$$hD^2_t\varphi + D^2_ts = -\gamma s - aD_ts.$$

The solution of these equations will be of the form (using ϭ for the Neperian base and ∂ for the ratio of circumference to diameter):

$$\left.\begin{array}{l} \varphi = A_1 \text{ϭ}^{z_1 t} + A_2 \text{ϭ}^{z_2 t} + A_3 \text{ϭ}^{z_3 t} + A_4 \text{ϭ}^{z_4 t}, \\ s = B_1 \text{ϭ}^{z_1 t} + B_2 \text{ϭ}^{z_2 t} + B_3 \text{ϭ}^{z_3 t} + B_4 \text{ϭ}^{z_4 t}, \end{array}\right\} \qquad (1)$$

where z_1, z_2, z_3, z_4, are the roots of the equation

$$(l - h)z^4 + alz^3 + (\gamma l + g)z^2 + agz + \gamma g = 0,$$

where, for each subscript letter,

$$B = -\left(l + \frac{g}{z^2}\right) A,$$

and where four arbitrary constants are determined by the initial conditions.

The roots of the biquadratic equation are all imaginary, and may be written

$$z_1 = -\xi_1 + \eta_1 \sqrt{-1} \qquad z_3 = -\xi_2 + \eta_2 \sqrt{-1}$$
$$z_2 = -\xi_1 - \eta_1 \sqrt{-1} \qquad z_4 = -\xi_2 - \eta_2 \sqrt{-1}$$

Expressing the coefficients in terms of the real and imaginary parts of the roots, the equation becomes

$$z^4 + 2(\xi_1 + \xi_2)z^3 + (4\xi_1\xi_2 + \xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2)z^2$$
$$+ 2[(\xi_1^2 + \eta_1^2)\,\xi_2 + (\xi_2^2 + \eta_2^2)\xi_1]z + (\xi_1^2 + \eta_1^2)(\xi_2^2 + \eta_2^2) = 0.$$

If the terms in z^3 and z were neglected, that is, if a were neglected, the solution of the false equation so obtained would be as follows (where observe the varying sign of η_1):—

$$\text{False } z^2 = -\tfrac{1}{2}(4\xi_1\xi_2 + \xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2) \pm \tfrac{1}{2}(4\xi_1\xi_2 + \xi_1^2 + \xi_2^2$$
$$- \eta_1^2 + \eta_2^2)\sqrt{1 + 4\,\frac{4\xi_1\xi_2\eta_1^2 - \xi_1^2(\eta_2^2 - \eta_1^2) - \xi_1^2\xi_2^2}{(4\xi_1\xi_2 + \xi_1^2 + \xi_2^2 - \eta_1^2 + \eta_2^2)^2}}.$$

Now, in the actual case, η_2 will be at least 100 times η_1, ξ_2 will be quite large, and ξ_1 very small. We may therefore neglect the square of the fraction under the radical; and we have very closely

$$\text{False } z_1^2 = \text{false } z_2^2 = -\eta_1^2 + \frac{4\xi_1\xi_2\eta_1^2 - \xi_1^2(\eta_2^2 - \eta_1^2) - \xi_1^2\xi_2^2}{4\xi_1\xi_2 + \xi_1^2 + \xi_2^2 - \eta_1^2 + \eta_2^2}$$

$$\text{False } z_3^2 = \text{false } z_4^2 = -\eta_2^2 - \xi_1^2 - \xi_2^2 - 4\xi_1\xi_2 -$$
$$\frac{4\xi_1\xi_2\eta_1^2 - \xi_1^2(\eta_2^2 - \eta_1^2) - \xi_1^2\xi_2^2}{4\xi_1\xi_2 + \xi_1^2 + \xi_2^2 - \eta_1^2 + \eta_2^2}.$$

$$\text{False } z_1 = -\text{ false } z_2 = \eta_1\left(1 - \frac{1}{2\eta_1^2}\,\frac{4\xi_1\xi_2\eta_1^2 - \xi_1^2(\eta_2^2 - \eta_1^2) - \xi_1^2\xi_2^2}{4\xi_1\xi_2 + \xi_1^2 + \xi_2^2 - \eta_1^2 + \eta_2^2}\right)\sqrt{-1}.$$

We thus see that, by neglecting the resistance, we get for the value of z_1 a quantity which requires only a minute correction in order to give the imaginary part of the true z_1. The same thing is not true for z_3 and z_4. Now, η_1 is ∂ divided by the principal period of oscillation of the pendulum upon the flexible stand. This is the quantity which we wish to determine; the others have only to be known approximately for the purpose of calculating the small correction to this. The logarithmic decrement of the amplitude of oscillation of the pendulum in the unit of time, so far as it is due to internal friction, is the quantity ξ_1. After these two quantities have been approximately ascertained, we may approximate to the quantity $(\xi_2^2 + \eta_2^2)$ by means of the equation

$$(\xi_1^2 + \eta_1^2)(\xi_2^2 + \eta_2^2) = \frac{\gamma g}{l - h}.$$

Then, by eliminating a between the two equations

$$2(\xi_1 + \xi_2) = \frac{al}{l - h},$$

$$2[(\xi_1^2 + \eta_1^2)\xi_2 + (\xi_2^2 + \eta_2^2)\xi_1] = \frac{ag}{l - h},$$

we obtain ξ_2, and consequently η_2. The values so obtained must satisfy the equation

$$4\xi_1\xi_2 + \xi_1^2 + \xi_2^2 + \eta_1^2 + \eta_2^2 = \frac{\gamma l + g}{l - h}.$$

Before proceeding to the consideration of the elastic after-effect, I propose to apply the equations thus obtained to the calculation of the correction of the seconds' pendulum for the flexure of the stand, supposing the internal friction to be proportional to the velocity.

For the pendulum used by me we have the approximate values: —

$l = 1.00$; h (heavy end up) $= 0.30$; h (heavy end down) $= 0.70$; g (New York) $= 0.993 \times \partial^2 = 9.89$; $\gamma = \frac{1}{0.0002125} = 4706$; $\eta_1 = 1.00$.

The accompanying table shows that $\xi_1 = 0.000008$. From this, we calculate that with heavy end up $\xi_2 = 0.08$, $\eta_2 = 257$; with heavy end down $\xi_2 = 0.17$, $\eta_2 = 392$. From this, it appears that the correction of η_1 is absolutely insensible, or, in other words, the effect of resistance (supposed proportional to the velocity) vanishes. That this is nearly, in fact, the case for my instrument is shown by the circumstance that the times of oscillation upon stands of different rigidities agree with the values calculated in leaving the internal friction out of account.

U. S. Coast Survey. Pendulum. Decrement of Arc due to internal friction of brass of tripod. Pendulum was swung on brass tripod in Paris, Geneva, and Kew. On a stand ten times as stiff in Hoboken. The times of decrement given are the SUM *of the times with the heavy end up and heavy end down.*

Half amplitude.	Time decrement on		Time shortened by internal friction.	Ratio of shortening.	Decrement in one second.	Decrement due to internal friction in one second.	Mean arc.	Natural logarithmic decrement due to internal friction.
	Flexible stand.	Stiff stand.						
100′								
	1073^s	1095^s	+ 22^s	.022	0′.0186	.00023	90′	.0000025
80								
	706	762	+ 56	.080	0 .0142	.00114	75	.0000152
70								
	1927	1969	+ 42	.020	0 .0104	.00037	60	.0000062
50								
	1377	1254	Reject.					
40							Mean	.000008

The last interval is probably affected by an error in the graduation of the scale used on one of the stands.

M. Plantamour proposes to determine the effect of the internal friction of the pendulum-stand upon the correction for flexure, by means of the difference between the statical and dynamical flexure. He has made numerous observations, which, according to his own interpretation of them, would show that, if a pendulum be supported in a certain inclined position until the stand has had time to take its position of equilibrium under this force, and then be let go, the ratio of the amplitude of oscillation of the stand to that of the pendulum is not the initial one, but is very different from that. If this were the case, the motion of the stand and pendulum could not be represented, even approximately, in the form (1), for by those equations the logarithmic decrement of the oscillation of the stand is the same as that of the pendulum. It is true that the two parts of the oscillation (nearly in the natural periods of the pendulum and of the stand) have different logarithmic decrements; and, as the ratio of their amplitudes is not the same for the stand and for the pendulum, a certain change in the total relative amplitude might occur in this way, but only an excessively minute one, nothing like what M. Plantamour thinks he has observed. But it is so improbable that the motions of the stand and pendulum depart much from the forms (1) that it would be wrong to accept M. Plantamour's results, until they are confirmed by a purely optical observation free from any possible influence from the machinery attached to the stand. Such an observation has been made by me; and, though I admit it was rather rough, it is entirely opposed to M. Plantamour's conclusions. Should the latter be confirmed, they would totally nullify the attempt to correct for the effect of flexure, as they would show the inapplicability of the analysis which has been proposed for the solution of that problem, without affording us much hope of being able to replace it; and it would seem to be necessary in that case to reject all the work which has been done with the reversible pendulum.

If the pendulum were started in the manner proposed, and if for any cause the amplitudes of pendulum and stand were altered in different ratios, there would be a perpetual force at work tending to restore the old ratio, so long as the phases of the motion were the same in the pendulum and stand. But, if the phases differed, a part of this force would go to diminishing the amplitudes, and would act so strongly in this way that there would be a rapid decrement on account of this circumstance. Suppose, for instance, that in the differential equations we were to put instead of $D_t^2 s$, $D_t^2 s_1$, where s_1 is the value of s at a time later than t by a constant. The result of this would be (neglecting terms involving a) that instead of the square of the exponent of the Neperian base being the sum of two negative quantities, one of them very small compared with the other, the smaller of these quantities would be multiplied by an imaginary root of unity. This would have but little effect on the imaginary part of the exponent of base, which determines the period; but it would add a considerable real part, which would represent a corresponding decrement of arc.

It seems difficult to conceive of a force which should greatly change the relative amplitudes of oscillation of the pendulum and stand, without at the same time producing an enormous decrement of the amplitude of oscillation, such as certainly does not exist. It is for those who believe that the existence of such a force has been experimentally proved to show how great an effect it would have upon the period of oscillation. *M. Plantamour supposes that the formula given by me in* my paper, "De l'influence de la flexibilité du trépied sur l'oscillation du pendule à reversion," would still apply to such a case; but I am unable to see upon what ground.

Meantime, in the present state of the question, it appears to me that we must appeal to direct experiment to determine the difference between the time of oscillation on a stiff and on a flexible stand. Such experiments were given by me in the paper above mentioned, and I have since greatly multiplied experiments on a stiff stand, with the general result there announced, namely that the difference is slightly greater than my theory supposes (owing, perhaps, to neglecting the energy of movement of the support), and not smaller, as M. Plantamour's views would require.

XXVIII.

COLOR-PERCEPTION.

BY G. STANLEY HALL.

Presented March 14, 1878.

THE finest distinctions which the ear can make — whether in detecting differences as small as one sixty-fourth of a whole note, or between harmonic upper partial tones — are, at first, purely mechanical processes of the terminal apparatus of the auditory nerve. Only after this process is complete, does the neural process of the extremely specialized fibres, which ends in the sensation of tone, begin. The physicist can follow the sound waves as they are conducted through the outer media of the ear, as their amplitude is diminished and their force increased; can calculate the amount of sympathetic vibration which will be caused in each part of the organism, as it passes; and at last determine the formulæ by which it is analyzed into pendular vibrations, by the organs of Corti or the fibres of the basilar membrane: thus tracing it to the very verge of consciousness itself.

In turning to the perception of color, we find that, if we take into account the analogies suggested by the undulatory theory, and the greater minuteness of the waves of the light-ether, the eye — so far as explored — responds to external stimulation with far less special mechanical adaptation than the ear. Masses of white light are thrown upon the retina, the focal distance of the various colors of which it is composed differing by twice the whole retinal diameter and six times the length of the longest rods, — every outline surrounded by dispersive and diffractive fringes, of ten times the diameter of the base of one cone; and here, with a vaguely defined, and to a great extent unverified, suggestion of three species of percipient elements, it is left.

If this were really all, and neural action had to collect the material of visual sensation in this form, and the mind, reacting upon such aggregate stimuli, were able to project the whole visible universe of color, — while we might understand why sight should have been the favorite sense of the Spiritualistic philosophy, we should, at the same time, be compelled to admit that the eye is a somewhat clumsy organ. If the ultimate fibres of the auditory nerve had been supposed to be directly sensitive only to the vibrations of the fluid of the labyrinth, and if the functions of the *ductus cochlearis* and all its exquisite mechanism — which does instantly what the mathematician only lately learned to work out by Fourier's intricate formulæ — had been undiscovered, the explanation of the ultimate processes involved in the sensation of hearing would be scarcely more satisfactory than those of the sensation of color-perception now are.

Purkinje, Volkmann, Helmholtz, and others have found that, if two parallel fibres of spider's web, or two fine wires, be brought very close together upon a white ground, the intermediate white line seems to have a beaded, or zigzag, outline, when closely examined with one eye.

Assuming the cones to be arranged somewhat in the form of hexagonal cells in a honey-comb, this has been explained by supposing that the retinal image of such a line is so small that, as it falls across this musive surface, one minute section of it would excite only one cone, while the sections immediately above and below would cover halves of two adjacent cones, and, exciting both to activity, would appear twice as large.

Now, if the ultimate percipient elements be cones of three varieties of sensibility, corresponding to the colors red, green, and blue, or violet, it follows that the cones which perceive, *e. g.* green, must be much more widely dispersed over the retina, being at most only one-third of the whole number; and hence, with black lines upon green ground, or the reverse, we might fairly expect this beaded irregularity to be much greater than if all the cones were excited, as they would be by white lines of light. To test this, I gummed ultimate fibres of white silk upon a smooth piece of heavy black paper, as near together as I could distinguish them when the lens under which I worked was removed. By bending the paper gently backward, the fibres were drawn tense and straight. After some practice, in the morning, when the eyes were fresh, or even after closing them for a minute, later in the day, I could distinctly recognize that the outlines of the white fibres were wavy, and even beaded. Now, as near them as possible, parallel fibres of bright red, green, blue, and violet were fastened, and viewed in a similar manner, under stronger illuminations. The same wavy outlines were observed, though with somewhat greater straining of the eyes. It is quite certain that the curves were no larger, and no less frequent, than with the white fibres. Thus, if the cause assigned to these appearances be the true one, and if my observations be verified, it would seem that the hypothesis of three sets of cones must be abandoned.*

*If, then, we do really reach the ultimate possible limits of surface-*perception without approaching one step towards the analysis of white light into its elementary colors, the only remaining hypothesis seems to be that they are distinguished in different planes of the retina. This, indeed, is countenanced by the calculation of Helmholtz that, when the eye is accommodated for a white object at convenient distances, the focus of the violet rays is .434 mm., or twice the thickness of all the layers of the retina in front of the focus of the red rays. Indeed, on the undulatory theory, the difference of wave-lengths must itself be a function of perception, which cannot therefore take place in a mathematical plane. Analogies of light and sound at once suggested to me sympathetic vibrations of the minute segments of the retinal cones, whose diameters, according to Max Schultze's measurements, are about the same as those of the wave-lengths near the red end of the spectrum. Accordingly, the following series of observations on positive, or incidental after-images, which have never been very fully investigated, was made:—

A series of bright-colored pieces of paper, four inches square, were fastened to a long strip of pasteboard, in the order in which they occur in the spectrum: a movable slit, of somewhat less diameter than the squares of paper, allowed any color to be seen by itself, without the effects of contrast. Positive after-images of each of these colors were formed, first by opening the eyes as suddenly and closing them again as quickly as possible, about once in a second, eight or ten times,—an experiment which was afterwards varied by illuminating the squares in a dark room by an electric spark, and later by observing, in the same manner, a solar spectrum, cast by two prisms of rock salt.

Beginning at one end of the spectrum, and trying each color successively, it was observed that, near the middle of the spectrum, the first phase of the positive after-image is nearly or quite white.

To my own eye, this effect is somewhat greater with highly saturated blue than with green, which appears dazzling white even beside white paper. No noticeable paling of red or of violet could be observed. The experiment was afterward varied by getting a positive after-image of the whole of a short spectrum at once, the middle of which still seemed nearly or quite white. Thinking that this might be due to the greater intensity which spectra formed by most prisms seem to have to the normal eye near the middle, these spectra were thrown upon red and violet paper, which absorbed most of the green rays; but still the effect was the same.

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* Of course, the purest colors obtainable in silk contain the whole spectrum, and the only precaution adopted was that of tiring the eye for the complementary color before fixating the fibres. Of ten High School boys, who were induced to try the experiment for several consecutive days, and, without being told what was expected, to draw the lines as they appeared, three represented them as wavy, and could observe no difference in the size of the curves in the white and in the colored fibres.

Now, if we suppose a series of sensitive elements — let us say disks, like the rattles on the tail of a snake — at the ends of the cones, each responding by sympathetic vibrations, or otherwise, to the action of waves in the light-ether, of corresponding length, the perception of white light would require the simultaneous agitation of all, or at least of several, groups of these disks. Let us assume also, for the present, that these disks are arranged in a spectral order, — those sensitive to red near the point, those sensitive to violet near the base, of the cone, — each disk being transparent to all waves of greater length than those to which it is best fitted to respond. If this were the case, agitation of a group of disks near the middle of what we will call the cone-spectrum might mechanically agitate the groups on either side, and thus give rise to a wave of disturbance, which, passing to both ends of the series, would cause a sensation of white, which the agitation of either end would not do to any such extent.

Again, it is well known that pressure, either mechanically applied, or caused by retinal congestion, often causes pure colored as well as white images. This has never been satisfactorily explained on the hypothesis of three sets of cones, or rods. So far as the effects of pressure have been observed on retinal purple of fresh eyes, the effect is always the same. If we assume that increasing degrees of pressure excite waves of disturbance of increasing length, involving a larger number of disks, we can readily believe that the effects of fatigue, determining the degree of instability of different segments along the cone-spectrum, would account for the various color-sensations thus produced.

Instead of three species of terminal organs, the modern form of Young's hypothesis assumes three sets of sensitive fibres, responding to the irritations of the three ground colors. The facts of red blindness afford, perhaps, the strongest ground for this theory. According to our hypothesis, however, red is perceived in the outer plane of the retina at the end of the cones. If we consider the delicacy and comparatively exposed position of these red disks among the coarser pigment cells of the choroid, and especially if we admit it to be proven that the two change their relative position with every irritation, we should expect that the ends of the cones would be often injured or undeveloped, as, indeed, the microscopist often finds them. How far the insensibility of the normal eye to the less refrangible red rays is due to the limit of retinal function, and how far to the absorptive power of the lens and humors, has never been determined. We should expect, however, to find some of the thermal rays just at or beyond the point of the external cone, perhaps limiting, and sometimes even impairing, its functions. Toward the violet end of the spectrum, there is good reason to believe that no solar rays enter our atmosphere which do not cause, at least, fluorescent sensibility in the retina.

Professor Fick has lately reinvestigated the facts of the color-blindness of the equatorial tracts of the retina, and frankly admits that the phenomena he has observed cannot be explained by the absence of a fundamental color. He endeavors, however, to preserve the hypothesis of three sets of fibres by arguing that the physical constitution of the eye is such that excitation must be considered as a function of oscillation. Thus, the longer ordinates of the curves representing the maxima of each of the three sensations in the spectral series of colors approach each other as the color-blindness here becomes complete, or as the angle of vision increases: so that, *e. g.*, a red ray, falling here, might appear yellow. Now, by assuming near the *ora serrata* either a shortening or an inclination of the cones, so that either the red disks are absent, or not reached by their corresponding rays, these phenomena, if we take into account the chromatic aberration of the ante-retinal eye, which Fick has entirely disregarded, can all be explained in a much simpler way. More accurate observations, however, than have yet been recorded, respecting the angle of inclination of the ray to the cone, and the amount and uniformity of shortening of the external cone near the front edge of the retina, are here needed.

The peculiar relation of green to the two other colors, as shown on the leverage curve, or chart of mixing, as reconstructed by Maxwell, has never been satisfactorily explained on the hypothesis of Young. Why have we here a curve, and not an angle, as at red and violet? or, in other words, why does the mixture of any two tones of green cause such a sudden and exceptional decrease in saturation?

When we consider the almost perfect integrity with which the green rays reach the retina, and its sensitiveness to them, we should expect not only a greater saturation than is observed in the spectrum, but also more shades and more distinct hues than we find upon the color table. The complementary hues of green are numerous and pronounced.

These unexplained facts, however, are very simply accounted for, when we reflect on the instability which the central position of the green disks would give them, and the readiness with which a wave of disturbance, starting here and passing each way, would produce the impression of an admixture of white light; while the abruptness with which the impression of green fades out, after the stimulus ceases, leads us to believe, according to the law of acoustic sensibility in sonorous bodies, that the *green disks give sympathetic response to a greater variety of wave-lengths than the red, or even the violet;* in other words, that the sympathetic function at the centre of the cone-spectrum is less specialized than at its ends. This, too, if rays are brought to a second focus in each cone, we should expect.

Passing to violet, we must believe that the retina is directly sensitive to its own fluorescence. This, Helmholtz says, is improbable. But, if fluorescence is a property of the anterior layers of the retina, why should the eye not be sensitive to it, as it is to the retinal blood-vessels, or to the almost constant stimulation of blood-color from other adjacent membranes? If, on the other hand, the light green which has been observed in a fresh retina, under the stimulation of ultra violet rays, is due to a complementary activity of the green disks, then, of course, the mind perceives it directly in the lavender gray which may be seen by a sensitive eye among the most refrangible rays. Of course, this question is greatly complicated by the fact that the lens is far more fluorescent, and scatters light blue rays all over the interior eye; and that even the cornea and vitreous humor aid in this general dispersion. While this would impair the distinctness of every violet image, it does not seem sufficient, without adding the effects of retinal fluorescence, to account for the evanescent touch of green which appears in ultra violet rays. It is possible, though hardly probable, that retinal fluorescence is a part of the function by which complementary colors are developed.

Before the late observations of Boll and Kühne, the theory that colors are to any extent reproduced in the eye would have been thought as baseless as the scholastic doctrine of visible species, or the Cartesian theory that colors are different rates of vortical motion. Boll now, however, not only insists that we must choose between what he terms the interpretation and the identity theories, but gives his vote for the latter. The great interest which the psychologist feels in these investigations is because this question is at least involved here. While it would be premature to say that all the observations thus far can be interpreted upon the hypothesis of a cone-spectrum, it seems quite certain that they cannot be interpreted in accordance with Young's theory.

It is, of course, yet possible that retinal purple is merely protective when the retina is most sensitive; viz., after rest. It is possibly connective tissue between the disks. It is, again, quite possible that its changes are connected with the perception of light and shade, and not at all with that of color. On the one hand, the fact that pressure causes the same sort of bleaching as light; that the color is chiefly confined to the external members of the outermost retinal layer, while near the vitreous body the cones and rods are nearly colorless; the fact that fluid solutions of retinal purple can be made only by a substance which causes the disks to fall apart, in Kühne's language, like a roll of coin suddenly unfastened; its high refractive power, — all these facts point decidedly toward the photo-physical explanation. On the other hand, we must not forget that the fact that retinal purple can be dissolved and filtered; the fact that perception probably takes place long before any sensible bleaching of the retina occurs; that colors are represented only by more or less paling, and never by changes of hue in the red substance; that it seems to be confined to the rods, and that pigment cells and the colored oil drop in the base of the outer members seem to undergo concomitant changes, perhaps regenerative, perhaps participatory; the fact that vinegar changes the red to instant yellow, — these facts, while they do not seem, as Kühne urges, to compel the belief that the retina is a photo-chemical workshop, or that the red is independent of all structural changes, do show us that we are here on the boundary line between chemistry and physics, and that the interpretation of each may be partial. Observations on retinal purple, and the fact that continued pressure on one eye causes all objects for a time to seem tinged with violet, would incline us to believe that the action of light on the cone-disks causes increased tension, possibly shortening instead of, or more probably along with, sympathetic vibration.*

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* On the photo-physical hypothesis, the almost constant action of red rays from the blood would tend to relax tension, — perhaps bringing a large number of disks into distances from each other corresponding to the length of the red instead of to that of the shorter waves, — and the retinal purple might then be due to the same cause as the color of thin plates; while the white which occurs with protracted illumination of a fixed eye would be physiologically analogous to tetanus; and the observer would see all the colors superimposed, producing the impression of white. This, again, would indicate that the phenomena of fatigue are to be explained, in part at least, by nervous exhaustion, and not entirely by failure of mechanical response in the terminal apparatus; and thus we are brought, by a very direct, scientific path, to the old question of phosphorescent effects in the eye.

Volkmann, Goethe, Brewster, and many others, have claimed to be able to distinguish with one motionless eye two component parts of a mixed color. Helmholtz urges that this is always a matter of judgment, and not of sensation. We are used, he says, to seeing things in the changing lights and shadows of morning, noon, and night, and by artificial illumination; and so unconsciously acquire the ability to judge what is due to the medium, and what comes from the object; and thus can often do so correctly the first time, when thin, white paper is spread over a colored surface. Wundt urges against Zenker's ingenious theory, that we ought to see white light as mixed. Now, in the first place, it is by no means proven that the eye cannot be trained to see one color through another at a distance from it, — which is the most favorable form of mixing; but even here each shade of a composite color may have a different local sign, while, as in our hypothesis, the elastic mechanical action of disks on each other — lying, as they do, so close together that no microscope can distinguish them in a fresh retina — prevents distinct perception. Investigation is much needed here. Meanwhile, we must not forget that the retina is not the only surface where different modes of irritation coalesce in a single sensation, and that the difficulty may be solely due to inadequate cultivation of attention and discrimination.

Our theory thus involves a redistribution of the causes of many well-known phenomena of vision. We must distinguish, first, those due to direct action of light on sensitive disks; second, those due to the action of these disks on each other; and, last, those due to the neural action thus occasioned: and the problem now before us is to determine how far such phenomena as after-images, contrast, persistence of impression, &c., are due to each of these influences. Such familiar facts as that change of brightness causes change of hue; that pale and dark colors are more contrasted than those that are full; that greater intervals intensify and emphasize colors, while small differences in tone seem most contrasted, — would indicate that contrast is not, as has been lately claimed, all a matter of subjective judgment, but that it may be due, in part, to the second of these causes. Complementary after-images, it is quite certain, cannot be explained by any theory of musical harmonics; for the musical fifth would give us a greater, the third a less, than the complementary interval on the color scale. We should expect, if the disks respond by vibration, that their motion would be far less rapid than that of light waves, corresponding perhaps as a very small multiple, or as a lower harmonic or difference tone; or, again, there may be one curve for the elasticity of disk action, or of the tissues connecting the disks, and another different curve representing neural fatigue; so that there would be two waves of excitation along the cone-spectrum, each followed at different intervals by slower, damping waves of fatigue: and it seems quite possible that the combined effects of both must be estimated in explaining all species of after-images. Many circumstances, which will readily occur to those who have followed us thus far, would lead us to believe that persistence of impressions would be found due more to the terminal apparatus than to the nerve action. The functions of the rods still remain unexplained. It has been suggested that, as white light is analogous to noise, the functions of the rods correspond to what was, until lately, supposed to be that of the vibratory hairs or the otoliths of the labyrinth. The very fact that these disks are smaller and more numerous and more uniform in size, while they contain apparently less nervous elements, suggests indeed that their function is to emphasize position more minutely than is done by the cones. That they do not perceive color is made to some extent probable by the fact that retinal purple, which seems to exist only in them, is not specifically modified by color as such; while, finally, if there is any reason to believe, as has been suggested, that cones are modified or developed rods, then the surprising theory of Hugo Magnus, which is so strangely countenanced by philological facts, — viz., that the color-sense has been developed out of simple perception of light and shade within the historic period, and in the spectral order, beginning with red, and with some corresponding loss in the accuracy of form-perception, — has at least one physiological fact in its favor.

Many other possible but unverified analogies with the ear are suggested: *e. g.*, may there be found any such correspondence between the length or number of cones and the brilliancy of colors developed among lower animals — and especially birds — by sexual selection as has been observed between the length of the *ductus cochlearis* and the development of songs and love-calls? Is there any special reason why, as the lower notes affect only the vibratory fibres most remote from the entrance of the labyrinth, so the red rays pass the other sensitive disks, to be perceived at the further end of the cone?

It is high time to remember that our theory can be called, at most, only probable, until the course of the rays through the retina can be more accurately traced. First, let us suppose light to undergo a special and final refraction, in the substance of the retina itself, before it is perceived. This is rendered highly probable by the great refractive power of the cone substance and of the retinal purple; by the fat globules and lentiform bodies observed in the cones of birds and reptiles; and by the bright appearance of the points of the cones, as seen from the back of an illuminated retina. This would be necessary, if rays of each color were to be focused on their appropriate disks. The distance between the focus of red and that of violet, as formed by the lens and humors, would then have to be reduced to, let us say, one fifteenth its estimated extent. This, again, would require such a dispersive surface as would bring rays, coming to different foci from different directions, into approximate parallelism; and, finally, a very strongly converging, and at the same time dispersive, power. Both these problems are theoretically solvable by the calculus of geometrical optics; but there still remains some doubt whether such a surface as is required in the first lens could really exist, and whether any known substance combines the high degree of refractive and dispersive power requisite for the second. It is, of course, conceivable that a substance might exist which should refract rays of one end of the spectrum, and transmit those of the other unaltered; and it is, again, very probable that each disk has a refractive power of its own. But, fortunately, none of the above suppositions are necessary. We may suppose that each disk responds to its own ray, from whatever direction it comes. In that case, the tardy action of red rays in producing their maximum sensation might be explained, in part, by the smallness of the red disks.

It is by no means necessary, as was assumed by the critics of Zenker's theory, that the thickness of the disks should in any way correspond to the length of the waves of light, or even that the phenomena of interference should be invoked. The difference may lie solely in size; it may depend on the angle of incidence; or, again, there may be such a special sensibility among the disks, and such minute accuracy in the refractive apparatus, that every red ray is thrown directly upon the sensitive point of its own disk, and that even another ray thrown there would seem to have the hue corresponding to the disk. All we want is a ponderable ether; while the fact that millions of vibrations are necessary before the faintest trace of color can be perceived sufficiently indicates its tenuity as compared with the medium of sound-waves.

The number of disks must be, at least, several hundred. Aubert was able to distinguish one thousand hues in the spectrum. Goblet and Rood believe we can learn to distinguish many million hues and shades. So it is probable that we shall here have to resort to the same explanation which Helmholtz gives of the fact that musicians can perceive one sixty-fourth of a semi-tone, which is less than the difference between the pitch of two of Corti's arches. In the one case, the two disks, as in the other the two arches, nearest attuned to the wave-length, would be excited, — the nearest one the most strongly; so that we shall never be able to perceive the steps or jumps which correspond to the difference between the ultimate percipient elements in either the tone or color scale.

The histologist Hensen long ago conjectured that perception took place at the outer segment of the cone, because the field of vision is made up of points so widely separated in comparison with their diameter. It has more recently been urged that the cones are not nervous substances, and that no fibres can be found running from their external segments. To this it can only be said that, if the size of the fibres bears any proportion to the length of the wave, observations on the ultimate nervous elements of the ear would teach us that comparatively large bundles of ultimate retinal fibres would still be invisible under the highest microscopic powers. The two hundred and fifty thousand fibres of the optic nerve may themselves be very complex. Our theory only adds a third to the two yet unexplored intervals along the diameter of the retina, which all who believe that the cones share in the act of visual perception admit must be somehow traversed by sentient processes.

If, then, certain phenomena, like colored shadows, or the subjective light of the closed eye, and many others, may be partly explained by the action of disks on each other; especially, if colors shall be found, in any sense, to be reproduced on the retina, — it may be that some of the many ingenious but mistaken color theories which have abounded in the world ever since Plato's day may have something to teach us which has been overlooked; for instance, Schopenhauer's, which has at least the merit of being purely physiological, while his formulæ of divisible remainders, and of qualitative and quantitative retinal activity, may be applied at once.

Chodin's observation, that moderate pressure always excites a sensation of green, whether on a black or white ground; Wheatstone's discovery, that on a smooth surface of red and blue squares the red seem raised; the fact that a distant landscape, viewed with inverted head, seems more like a flat surface, but more brightly colored, so that color and the third dimension of space seem reciprocal functions, — these and many other observations seem more or less fully explained by our hypothesis, which at least suggests new paths of investigation, and may perhaps even justify the question how much better, at this stage of our knowledge, we are really fitted to decide whether a color-scale is possible, than musicians who lived when a three-stringed harp was the most perfect instrument were to discuss diatonic intervals.

The writer desires to express his unusual obligations to Professor H. P. Bowditch for suggestion, criticism, and supervision of laboratory work, and to Dr. B. Joy Jeffries for the free use of his valuable library.

XXIX.

CONTRIBUTIONS FROM THE PHYSICAL LABORATORY OF HARVARD COLLEGE.

No. XIII.—ON THE INTENSITY OF TERRESTRIAL MAGNETISM AT CAMBRIDGE.

By Henry Goldmark.

Presented April 10, 1878.

As the intensity of the force of the earth's magnetism has not been determined in Cambridge for many years, it was thought that a measurement of its magnitude might prove of some interest. I measured only the horizontal component and the inclination, and deduced the value of the vertical component from these.

To obtain the *horizontal component*, I made use of Gauss's method of oscillations, using the torsion balance made for this purpose by Edelmann, of Munich. Two quantities are determined, the product MH of the horizontal intensity H and the magnetic moment M of the magnet used, and the ratio $\frac{M}{H}$ of these two quantities.

To get MH, the time of oscillation of a small cylindrical magnet, suspended by a silk thread, was determined by means of a mirror and scale. The number of complete oscillations and fractions of an oscillation which this magnet made in one minute of time when vibrating under the influence of the magnetic force was observed, and from this the time t of a single oscillation was easily obtained. The amplitude of vibration was in every case so small that the usual reduction to an infinitely small arc was found to be unnecessary.

The *coefficient of torsion* Θ of the thread was in every case found by turning the upper circle through an angle of 90°, observing the angular deflection (φ) from the magnetic meridian produced in the suspended magnet, and substituting this value in the equation

$$\Theta = \frac{\phi}{90^\circ - \phi}$$

The small cylindrical magnet was carefully measured by means of a dividing-engine.

For the diameter, I found

North end.	Mean of five measurements	$d = 14.395$ mms.
South end.	„ „ „	$d = 14.399$ „
Giving as the mean diameter		$d = 14.397$ „

The length was given as the average of five measurements equal to 49.448 mms.

The weight was 62109.2 milligrams.

If, now, following Rankine, we use the weight instead of the mass in determining the moment of inertia, we get

if l = length of the cylinder and
r = its radius

$$I = W\left(\frac{l^2}{12} + \frac{r^2}{4}\right) = 62109.2\left(\frac{(49.448)^2}{12} + \frac{(7.198)^2}{4}\right)$$
$$= 13459758.5 \text{ mms. mgms.}$$

We have, then, all the data to get the value of MH which is given by Gauss's formula

$$MH = \frac{\pi^2 I}{t^2 (1 + \Theta)}. \qquad \text{(A.)}$$

To get $\frac{M}{H}$, I suspended a small magnet carrying a plane mirror from the centre of the apparatus, and measured the deflections produced by the cylindrical magnet described above, when placed at two points to the east and to the west. Using both poles in each position gives eight measurements. I also made a few observations with a box compass, but the results were by no means as precise or accurate as those obtained by suspension.

If, now, φ and φ' are the angular deviations from the meridian produced by the magnet at the distances r and r', and if

Θ = the coefficient of torsion of the thread, we have

$$\frac{M}{H} = \tfrac{1}{2}\,\frac{r^5 \tan\phi - r_1 \tan\phi_1}{r^2 - r_1^{\,2}}(1 + \Theta); \qquad \text{(B.)}$$

and we easily see that

$$H = \sqrt{MH \div \frac{M}{H}}.$$

The *inclination* or *dip* was determined according to the method given by Weber (Poggendorff's Annal. XC.), by the strength of the inductive currents produced in a coil of copper wire, rotated in such a way that only one of the two components of the earth's magnetism acts upon it at one time. I first placed a coil vertically in the meridian and turned it 90° around a vertical axis, having connected the ends of the wire with a Thomson reflecting galvanometer. The horizontal lines of magnetic force are the only ones cut by the coil, and hence the scale reading is proportional to H.

In the next place, I turned the coil, again placed vertically on the meridian, 90° around an horizontal axis. In this case, the vertical component is cut, and a deflection proportional to it produced.

Therefore, we have found the ratio of the two components; and, if i = the inclination, we have

$$\tan i = \frac{V}{H}; \qquad \text{(C.)}$$

The results of my measurements are given below.

PRELIMINARY MEASUREMENTS, DEC. 19 AND 20, 1877.

(α) *Determination of MH.*

To find the time of oscillation t, two measurements were made as follows:—

Number of Observation.	Number of double oscillat. per minute.
I.	4.5
II.	4.509
	Mean 4.5045
which gives $t = 6.66$ secs.	

Θ was found equal to .01387.

Substituting these values in equation (A.), we get

$$MH = 2954000.$$

(β) *Determination of* $\frac{M}{H}$.

In this case, a ring-magnet was used, carrying a plane mirror, and suspended by a piece of silk thread, the torsion of which was measured, and gave

$$\Theta = .01719$$

The distance of the mirror from the scale was 1797 mms.

$r = 277$ mms.		$r_1 = 366$ mms.	
Position of deflecting magnet.	Deflection in scale divisions.	Position of deflecting magnet.	Deflection in scale magnets.
(1)	359.5	(5)	155.
(2)	354.	(6)	156.5
(3)	353.5	(7)	157.
(4)	359.5	(8)	152.5
	Mean 356.625		Mean 155.25
which gives $\phi = 5^\circ\ 36'\ 45''$		which gives $\phi' = 2^\circ\ 28'\ 8''$	

Substituting these values in (B.), we get

$$\frac{M}{H} = 1092495\,;$$

and combining with the value of MH,

$$H = \sqrt{\frac{2954000}{1092495}} = 1.6444$$

(γ) *Determination of the Dip.*

Six measures for each component were made, with the following results: —

HORIZONTAL COMPONENT.		VERTICAL COMPONENT.	
Number of Observation.	Deflection on the scale.	Number of Observation.	Deflection on the scale.
I.	23	I.	86
II.	22	II.	88
III.	23	III.	88
IV.	23	IV.	87
V.	22	V.	87
VI.	23	VI.	87
	Mean 22.66		Mean 87.16

Hence

$$\tan i = \frac{87.16}{22.66};$$

which gives

$$i = 75^\circ\ 25'\ 30''.$$

MEASUREMENTS OF MARCH 20, 1878.

(α) *Determination of MH.*

To find t, 8 measurements were made, with the following results: —

Number of Observation.	Amplitude of Oscillation.	No of double Oscill. per min'te.	t.
I.	174	4.5581	6.5816
II.	147	4.4578	6.5821
III.	99	4.5606	6.5781
IV.	90	4.5556	6.5853
V.	67	4.5606	6.5781
VI.	61	4.5590	6.5804
VII.	52	4.5577	6.5823
VIII.	34	4.5583	6.5807
			Mean 6.5811

Θ was found equal to .01979;

and these values give

$$MH = 3005600.$$

(β) *Determination of* $\frac{M}{H}$.

I here used a very light magnet, consisting merely of three pieces of magnetized watchspring. A mirror and a piece of aluminum foil to stop the vibrations were used. It was suspended by a single fibre of silk about 400 mms. long. The torsion of this thread was imperceptible even on turning the upper circle through an angle of 360°.

Six complete measurements were made, with the following results. The distance of the scale from the mirror was 1516 mms.

$r = 277$ mms.			$r_1 = 366$ mms.	
Number of Measurement.	Position of Magnet.	Deflection in scale divisions.	Position of Magnet.	Deflection in scale divisions.
I.	(1)	308	(5)	136
	(2)	312	(6)	135.5
	(3)	311	(7)	135
	(4)	309	(8)	132.5
II.	(1)	311	(5)	135
	(2)	313	(6)	135
	(3)	311	(7)	136
	(4)	309	(8)	133
III.	(1)	312	(5)	135
	(2)	311	(6)	135
	(3)	313	(7)	135
	(4)	309	(8)	133
IV.	(1)	311	(5)	135
	(2)	312	(6)	135
	(3)	312	(7)	134
	(4)	309	(8)	133
V.	(1)	312	(5)	134.5
	(2)	310	(6)	135.5
	(3)	312	(7)	135
	(4)	311	(8)	133
VI.	(1)	310	(5)	135.5
	(2)	311	(6)	135
	(3)	311	(7)	135
	(4)	309	(8)	133
		Mean 310.7916		Mean 134.5625
which gives $\phi = 5^\circ\ 47'\ 34\frac{1}{2}''$			which gives $\phi' = 2^\circ\ 32'\ 10''$	

Substituting these values, I find

$$\frac{M}{H} = 1096016\,;$$

and hence

$$H = \sqrt{\frac{3005600}{1096016}} = 1.656.$$

(γ) *Determination of i.*

I used two different coils, with results that agreed very nearly, as shown by the table which follows:—

	Smaller Coil *A.*		Larger Coil *B.*	
Number of Measurement.	*H*	*V*	*H*	*V*
(1)	45	10	86	22
(2)	44	12	87	22
(3)	43	10 5	85	22
(4)	45	12	86.5	21
(5)	44	11.5	87	22
(6)	43	12	87	23
	Mean 44	Mean 11.33	Mean 86.42	Mean 22

From *A* we get, then,

$$\tan i = \frac{11.33}{44} \qquad i = 75^\circ\ 33'\ 22''.$$

From *B* we get

$$\tan i = \frac{22}{86.42} \qquad i = 75^\circ\ 43'\ 3''.$$

The mean of these two values gives

$$i = 75^\circ\ 38'\ 12\tfrac{1}{2}''.$$

As our result, we have, then, for the force of magnetism on the 20th of March,

$$H = 1.656 \text{ mms. mgms.}$$

and

$$i = 75^\circ\ 38'\ 12\tfrac{1}{2}''.$$

And since

$$V = H \tan i,$$

we get

$$V = 6.4173.$$

Note. — To ascertain, at least approximately, the degree of correctness attained in this result, I computed the probable mean errors of the time of oscillation in the last determination. The table explains itself.

Number of Measurement.	t.	δ	δ²
I.	6.5816	+ .0005	.00000025
II.	6.5821	+ .0010	.00000100
III.	6.5781	— .0030	.00000900
IV.	6.5853	+ .0042	.00001764
V.	6.5781	— .0030	.00000900
VI.	6.5804	— .0007	.00000049
VII.	6.5823	+ .0012	.00000144
VIII.	6.5807	— .0004	.00000016
	Mean 6.5811		$S = .00003898$

The mean error of one observation $= \sqrt{\frac{.00003898}{7}} = \pm .00236$

„ „ „ the result $= \sqrt{\frac{.00003898}{7 \times 8}} = \pm .0008344$

The probable error of one observ'n. $= .67449 \sqrt{\frac{.00003898}{7}} = \pm .00159$

„ „ „ the result $= .67449 \sqrt{\frac{.00003898}{7 \times 8}} = \pm .00056278$

I am indebted to Professor Trowbridge for much kind advice and assistance in the course of my work.

PROCEEDINGS.

Seven hundred and second Meeting.

May 29, 1877. — Annual Meeting.

The President in the chair.

The Corresponding Secretary presented the Report of the Council.

The Report of the Rumford Committee was read and accepted.

The President announced the death of Mr. Edmund Quincy, in the following words: —

"It has become my painful duty to announce to the members of the Academy the loss they have, since the last meeting, met with in the person of Mr. Edmund Quincy, who has for several years served us faithfully in the responsible positions of our Treasurer and Librarian. On this occasion, it is not for me to enter at large into the detail which contributes to form his character. That will be done elsewhere in its proper place. Mr. Quincy had no fancy for display. What work he did was done modestly but effectively. As a writer, few men in America have excelled him in beauty, simplicity, and accuracy; and, as a man, I trust it may be allowed me to testify from an experience now of more than sixty years that he has not left a better or more honest man behind him."

The annual election resulted in the choice of the following officers: —

Charles Francis Adams, *President.*
Joseph Lovering, *Vice-President.*
Josiah P. Cooke, Jr., *Corresponding Secretary.*
Henry P. Bowditch, *Recording Secretary.*
Theodore Lyman, *Treasurer.*
Samuel H. Scudder, *Librarian.*

Council.

WOLCOTT GIBBS,
JOHN D. RUNKLE,
EDWARD C. PICKERING,
of Class I.

BENJ. E. COTTING,
ASA GRAY,
HERMANN A. HAGEN,
of Class II.

ANDREW P. PEABODY,
SAMUEL ELIOT,
CHARLES E. NORTON,
of Class III.

Rumford Committee.

MORRILL WYMAN.
WOLCOTT GIBBS.
EDWARD C. PICKERING.
JAMES B. FRANCIS.
JOHN M. ORDWAY.
STEPHEN P. RUGGLES.
JOHN TROWBRIDGE.

Committee on Finance.

CHARLES FRANCIS ADAMS,
THEODORE LYMAN,
ex officio.
THOMAS T. BOUVÉ.

The following gentlemen were elected members of the Academy: —

Leopold Trouvelot, of Cambridge, to be a Resident Fellow in Class I., Section 2.

August Wilhelm Hofmann, of Berlin, to be a Foreign Honorary Member in Class I., Section 3, in place of the late Johann Christian Poggendorff.

Oswald Heer, of Zurich, to be a Foreign Honorary Member in Class II., Section 2, in place of the late Wilhelm Friedrich Benedict Hofmeister.

Rudolph Leuckart, of Leipzig, to be a Foreign Honorary Member in Class II., Section 3, in place of the late Christian Gottfried Ehrenberg.

Johann Japetus Smith Steenstrup, of Copenhagen, to be a Foreign Honorary Member in Class II., Section 3, in place of the late Karl Ernst von Baer.

Mr. S. Watson presented the following paper by title: —

"Descriptions of New Species of Plants with Synopses of Certain Genera."

Seven hundred and third Meeting.

June 13, 1877. — ADJOURNED ANNUAL MEETING.

The PRESIDENT in the chair.

The following letters were read by the Corresponding Secretary: —

1. From the Accademia Gioenia di Scienze Naturali in Catania, accompanying a medal struck in commemoration of the fiftieth anniversary of the Society.

2. From the Senatus Academici Upsaliensis, inviting the Academy to join in the celebration of the fourth centennial of the University.

3. From the Société Botanique et d'Horticulture de Paris, inviting the Academy to take part in the Congrès de Botanique et d'Horticulture to be held at Paris, during the coming Exposition, on the 16th and 22d of August, 1878.

On the motion of Mr. Lyman, it was

Voted, To appropriate, —

For general expenses	$2,100
For the Library	700

On the motion of Mr. Pickering, it was

Voted, That the thanks of the Academy be presented to the Academy of Catania for the medal commemorating the fiftieth anniversary of the Society.

Mr. Lyman presented the annual report of the Treasurer. Remarks on this report were made by Messrs. Cooke and Lovering.

On the motion of Mr. Pickering, it was

Voted, That the appropriations called for in the report of the Rumford Committee be made.

On the motion of Mr. Cooke, it was

Voted, That an appropriation of twelve hundred dollars ($1,200) from the general fund be made for the publication of the Proceedings for the past year.

Voted, That a provisional appropriation of fifteen hundred dollars ($1,500) be made for the publication of the Proceedings for next year.

Voted, That the Corresponding Secretary be authorized to accept, in the name of the Academy, the invitation of the University of Upsala.

The chair appointed the following committees: —

Committee on Publication.

ALEXANDER AGASSIZ. W. W. GOODWIN.
JOHN TROWBRIDGE.

Committee on Library.

EDWARD C. PICKERING. HENRY P. BOWDITCH.
WILLIAM R. NICHOLS.

Auditing Committee.

HENRY G. DENNY. ROBERT W. HOOPER.

The chair announced the death of J. Lothrop Motley, Resident Fellow.

The following papers were presented: —

"On the Photographic Action of Rays of Solar Light of Different Refrangibility on Dry Silver Bromide Collodion." By Dr. Robert Amory.

"On a New Method of Determining the Errors of Meridian Circles." By Professor E. C. Pickering.

"A Notice of some Experiments in Confirmation of a Previous Statement that the Periodic Errors in Micrometer Screws are due to the Mounting of the Screws, and not to the Screws themselves." By Professor W. A. Rogers.

Seven hundred and fourth Meeting.

October 10, 1877. — STATED MEETING.

The PRESIDENT in the chair.

Dr. H. P. Bowditch presented his resignation of the office of Recording Secretary. The resignation was accepted, and Professor John Trowbridge was appointed to fill the vacancy.

The President announced the death of M. LeVerrier of Paris, Foreign Honorary Member, and of John H. Temple, of West Roxbury, Resident Fellows.

Mr. C. E. Norton presented, by title, a paper on the Dimensions and Proportions of the Temple of Zeus at Olympia.

Mr. Scudder exhibited a fossil butterfly from the tertiary formation of Colorado.

The following gentlemen were elected members of the Academy: —

John Rodgers, of Washington, to be an Associate Fellow in Class I., Section 4.

Arthur Searle, of Cambridge, to be a Resident Fellow in Class I., Section 2.

Charles R. Cross, of Boston, to be a Resident Fellow in Class I., Section 3.

Amos E. Dolbear, of Somerville, to be a Resident Fellow in Class I., Section 3.

George Cheyne Shattuck, of Boston, to be a Resident Fellow in Class II., Section 4.

Francis Minot, of Boston, to be a Resident Fellow in Class II., Section 4.

Charles Smith Bradley, of Cambridge, to be a Resident Fellow in Class III., Section 1.

Oliver Wendell Holmes, Jr., of Boston, to be a Resident Fellow in Class III., Section 1.

John Lowell, of Boston, to be a Resident Fellow in Class III., Section 1.

James Bradley Thayer, of Cambridge, to be a Resident Fellow in Class III., Section 1.

The following papers were presented: —

"Note on Grassmann's Calculus of Extension." By Mr. C. S. Peirce.

"On a New Form of a Dividing-Engine." By Professor W. A. Rogers. A machine built for the physical laboratory of Princeton College was exhibited.

"On the Determination of the Chances at Billiards in the Case of a 'Grand Discount.'" By Professor Benjamin Peirce.

Seven hundred and fifth Meeting.

November 14, 1877. — ADJOURNED STATED MEETING.

The PRESIDENT in the chair.

Dr. Wyman, in behalf of the Rumford Committee, asked for an appropriation of one thousand dollars ($1,000) from the income of the Rumford Fund, to be expended under the direction of the Committee on investigations in light and heat; and this appropriation was made.

Professor Watson presented to the Academy a volume containing studies of certain inventions exhibited at the late Centennial Exhibition, and also a study of engineering works upon the river Marne in France.

On the recommendation of the Rumford Committee, it was voted that the barometer belonging to the Academy should be loaned to the Institute of Technology for meteorological investigations.

The following papers were presented: —

"On the Dimensions and Proportions of the Temple of Zeus at Olympia." By Professor Charles E. Norton.

"On the Possible Affinities of a Problematical Fossil from the Carboniferous Rocks of Illinois." By Mr. S. H. Scudder.

Professor Semper made some remarks on the inhabitants of the Pelew Islands.

Mr. Trouvelot presented, by title, the following papers: —

"Undulations observed in the Light of Coggia's Comet of 1874."

"The Moon's Zodiacal Light."

"The Sudden Extinction of the Light of a Solar Protuberance."

Professor Trowbridge presented, by title, a paper by Mr. B. O. Peirce, Jr., and Mr. Lefavour, "On the Law of the Propagation of Heat in Solid Bodies."

Seven hundred and sixth Meeting.

December 12, 1877. — MONTHLY MEETING.

The PRESIDENT in the chair.

Professor Pickering presented a bound volume of his "Researches in Physics," and read a paper on Atmospheric Refraction. He also presented, by title, the following papers: —

"Supplementary Note on the Theory of the Horizontal Photoheliograph." By Professor William Harkness.

"On a Method of Comparing Short Standards of Length." By Mr. Leonard Waldo.

Professor Peirce presented, by title, a paper on Peirce's Criterion for the Rejection of Doubtful Observations.

Professor C. L. Jackson presented a paper by himself and Mr. A. W. Field on Parachlorbenzylchloride and its Derivatives.

Seven hundred and seventh Meeting.

January 9, 1878. — STATED MEETING.

The PRESIDENT in the chair.

The chair announced the death of Dr. J. P. Kirtland, of Cleveland, Ohio.

On the motion of Professor Pickering, it was

Voted, To appropriate two hundred dollars ($200) for improvements in the library room.

The following papers were presented: —

"On the Prediction of Neptune." By Professor Benjamin Peirce.

"On the Theory of Absorption Bands, and its Bearing on Photography and Chemistry." By Doctor Robert Amory.

"On the Use of a Lens of Long Focus as a Collimator." By Doctor Robert Amory.

Professor Cooke presented, by title, the following paper: —

"On the Copper-bearing Rocks of Lake Superior." By Professor Raphael Pumpelly.

Professor Peirce presented the following papers by title: —

"On Surfaces of the Second Order as treated by Quaternions." By Abbott Lawrence Lowell.

"Spherical Conics." By Gerritt Smith Sykes.

"Investigations in Quaternions." By Washington Irving Stringham.

Professor J. B. Thayer presented to the Academy a volume of letters and remembrances of the late Chauncey Wright.

The following gentlemen were elected members of the Academy: —

George Clark, of Cambridge, to be a Resident Fellow in Class I., Section 2.

Thomas P. James, of Cambridge, to be a Resident Fellow in Class II., Section 2.

John Fiske, of Cambridge, to be a Resident Fellow in Class III., Section 1.

Charles Greely Loring, of Boston, to be a Resident Fellow in Class III., Section 4.

Ezekiel B. Elliott, of Washington, to be an Associate Fellow in Class I., Section 1.

Raphael Pumpelly, of Owego, to be an Associate Fellow in Class II., Section 1.

Charles Sanders Peirce, of New York, to be an Associate Fellow in Class III., Section 1.

Carl Nägeli, of Munich, to be a Foreign Honorary Member in Class II., Section 2, in place of the late Alexander Braun.

Seven hundred and eighth Meeting.

February 13, 1878. — ADJOURNED STATED MEETING.

The PRESIDENT in the chair.

The death of Victor Regnault, Foreign Honorary Member, was announced.

On the motion of Professor Peirce, it was

Voted, That a committee be appointed to memorialize Congress in regard to the position of the Astronomical Observatory at Washington, and that this committee consist of seven members, with the President of the Academy as chairman.

On the motion of Professor Gray, it was

Voted, That this committee report to the Academy.

The committee appointed consisted of Messrs. C. W. Eliot, J. D. Runkle, W. B. Rogers, J. Lovering, W. Gibbs, and B. Peirce.

The following papers were presented: —

"On Peirce's Criterion for the Rejection of Doubtful Observations." By Professor Benjamin Peirce.

"A Method for Demonstrating Gravitative Action between Small Masses." By Professor A. E. Dolbear.

"On the Aerial Respiration of the Amia." By Professor B. G. Wilder. (By invitation.)

"A Note on some Species of Uredineæ." By Professor W. G. Farlow.

Professor Gibbs read a paper by Dr. F. A. Gooch, "On a New Method of Filtering."

Professor H. P. Bowditch presented, by title, a paper, "On the Theory of Color-Perception." By G. S. Hall.

Seven hundred and ninth Meeting.

March 13, 1878. — STATED MEETING.

The PRESIDENT in the chair.

The chair announced the death of Elias Magnus Fries and Count Sclopis, Foreign Honorary Members.

Professor Lovering called attention to a former vote of the Academy, requesting the government to require postmasters to collect statistics in regard to persons struck by lightning, and presented the following report of a Committee appointed by the Academy to take this matter into consideration: —

"The discoveries and inventions in electricity, since the time of Franklin, have not added much to our knowledge of thunder and lightning, or of the best means of protection against them. While science is modest, if not altogether silent in this matter, there is no lack of loud talkers who trade upon the ignorance and fears of the public. The claims of different patentees are so conflicting that a thoughtful man may well doubt what he should do, and come finally to the conclusion that the safest course for him is to do nothing. A faithful record of accidents to persons and property by lightning (of which this large country would furnish numerous examples every year) with a detailed account of the exposure in each case, either with or without lightning rods, will put on trial old devices for protection, and may suggest new ones. In Europe, governments and academies have made it their duty to investigate this subject and to instruct and guide the public. The signal service at Washington, which has already done much good work for the community, and which has in charge certain questions in meteorology, might be able to enlarge its sphere of duties, and already possesses an organization well adapted to obtaining the information, the usefulness and necessity of which have already been indicated. Your Committee therefore recommend to the Academy the adoption of the following vote, — to be communicated, together with the explanatory remarks which have preceded, to the Chief of the Signal Service at Washington: —

"*Voted*, That the Chief of the Signal Service at Washington be requested to use such means as may be at his command for collecting and publishing full and accurate statistics in regard to accidents by lightning in the United States.

On recommendation of the Rumford Committee, it was voted to charge to the income of the Rumford Fund the papers in Vol. XIII. of the Proceedings of the Academy numbered V., VIII., X., XI., XII., XIV., and XVII., and

Professor Cooke made a report upon the funds for the publication of the Proceedings, and moved an additional appropriation of five hundred dollars ($500) for printing the Proceedings of this year.

The following gentlemen were elected members of the Academy: —

Edward Burgess, of Boston, to be a Resident Fellow in Class II., Section 3.

James Jackson Putnam, of Boston, to be a Resident Fellow in Class II., Section 3.

John Collins Warren, of Boston, to be a Resident Fellow in Class II., Section 4.

Phillips Brooks, of Boston, to be a Resident Fellow in Class III., Section 1.

John Williams White, of Cambridge, to be a Resident Fellow in Class III., Section 2.

Justin Winsor, of Cambridge, to be a Resident Fellow in Class III., Section 2.

Émile Plantamour, of Geneva, to be a Foreign Honorary Member in Class I., Section 2, in place of the late Urbain-Jean-Joseph LeVerrier.

Mr. S. H. Scudder presented a paper "On the Discovery of Insect Eggs in the Laramie Group of Rocks."

Professor B. Peirce presented, by title, a paper by Mr. C. S. Peirce, "On the Influence of Internal Friction upon the Correction of the Seconds' Pendulum for the Flexibility of the Support."

Professor B. Peirce made some remarks on the internal structure of the earth with reference to Lipswich's results in regard to its density, and the theory of Sir William Thomson.

Professor W. A. Rogers made some remarks on the measurement of standards of length.

Mr. Trouvelot exhibited the results of his late observations on Jupiter.

Seven hundred and eleventh Meeting.

May 8, 1878. — MONTHLY MEETING.

The PRESIDENT in the chair.

Professor Trowbridge exhibited a new induction coil.

Professor Dolbear exhibited a number of new radiometers.

Professor Searle showed some photographs of the transit of Mercury.

Professor Gibbs communicated the following papers, by title: —

1. On the Law of Boyle and Mariotte.
2. On a Compensated Air Thermometer.
3. On a Differential Calorimeter.

Professor Farlow communicated the following paper, by title: —

Contributions to the Marine Flora of the United States, No. 3.

REPORT OF THE COUNCIL.

SINCE the last report, May 9, 1877, the Academy has lost by death fifteen members, as follows: eight Fellows, George Bemis, George T. Bigelow, Edward H. Clarke, John Lothrop Motley, Charles Pickering, Edmund Quincy, John H. Temple, and John E. Tyler; two Associate Fellows, Joseph Henry, J. P. Kirtland; five Foreign Honorary Members, Fries, LeVerrier, Regnault, Thiers, and Count Sclopis.

GEORGE BEMIS.

GEORGE BEMIS was born at Watertown, Massachusetts, October 13, 1816, and died at Nice, in France, January 5, 1878. He was the son of Seth Bemis, a manufacturer in Watertown, from whom he inherited a good property. He graduated at Harvard College in 1835, and at the Dane Law School in 1839, and was admitted to the bar in the same year. "His legal training," it has been said, "was very thorough; and his learning, acuteness, diligence, and fidelity gave him very soon a good position at the bar and a profitable practice." He distinguished himself in several criminal cases, — especially as junior counsel, in the year 1850, at the trial of Dr. Webster for the murder of Dr. Parkman.

Ill-health compelled his withdrawal from practice in the year 1858. After that time, he travelled much in Europe, and pursued with interest the study of public and international law. During the War of the Rebellion, he made important contributions to the discussion of some of the principal questions relating to neutral and belligerent rights, and published several pamphlets on these subjects.

"Mr. Bemis was a man of singular purity and refinement of character. . . . He was never married; but was social, friendly, and hospitable, affectionate, and sincere."

He was a member of the Massachusetts Historical Society and of this Academy; and he left a legacy to each of these societies. To Harvard College, also, he left the sum of fifty thousand dollars, for the endowment of "a professorship of public or international law in the Dane Law School."

GEORGE TYLER BIGELOW.

George Tyler Bigelow was born at Watertown, Massachusetts, October 6, 1810, and died in Boston, April 12, 1878. He was a son of the Hon. Tyler Bigelow, an eminent lawyer in Middlesex county, and grandson of Colonel Timothy Bigelow of Worcester, an officer in the war of the Revolution. He graduated at Harvard College in the class of 1829, and afterwards pursued the study of law. In 1834, he was admitted to the bar, and established himself in the practice of his profession at Boston. He was interested in political affairs, and at different times in early life was a member of the Legislature of Massachusetts; but his main strength was devoted to the profession in which he was destined to win the highest honors that his native State could give. "At the bar," it has been said, "he was active, energetic, industrious, indefatigable, with plenty of courage and tenacity."

In 1848, he was appointed by Governor Briggs a justice of the Court of Common Pleas. The appointment was much criticised, but the criticism did not continue long. One who knew him well has said, "From the first day he took his seat, he was every inch a judge. In the despatch of business, in the management of the docket, in his clear and able charges to the jury, in his absolute impartiality, he won the applause and even the admiration of the bar."

In 1850, he was promoted to the Supreme Judicial Court as successor to Mr. Justice Wilde. As to the manner in which he filled this office, one of his associates, Mr. Justice Hoar, has said: "His learning and sagacity, his love of the law as a science, his readiness of apprehension, and, more than all, his wonderful power of appropriating, submitting to legal tests, and bringing to practical and safe results the ideas and suggestions of others, whether at the bar or from his associates on the bench, made him an invaluable member of the Court.

On the retirement of Chief Justice Shaw in 1860, Judge Bigelow succeeded him in the highest judicial office in the State. He held this office with honor until the year 1867, when certain physical infirmities led him to withdraw from judicial life. He accepted at once the office of actuary of the Massachusetts Hospital Life Insurance Company, and held that place until the beginning of the year 1878.

Judge Bigelow was a member of the Corporation of Harvard College from the year 1868 to the time of his death. He was also Vice-President of this Academy.

EDWARD HAMMOND CLARKE.

Edward Hammond Clarke was born in the town of Norton, Massachusetts, February 2, 1820, and died in Boston, November 30, 1877.

His father, the Reverend Pitt Clarke, born in 1763, graduated at Harvard College in 1790, was minister of the first Congregational Society in Norton for the long period of forty-two years, and died in 1835. He left a brief Autobiography and a "Confession of Faith," both of which are interesting, revealing as they do the simplicity and purity of his character and the manly clearness of his intelligence.

Mary Jones Clarke, the mother of the subject of this notice, was a woman in every way worthy of her most estimable husband. She joined to great excellence of character a remarkable poetic taste, and a talent which found expression in many pleasing domestic, devotional, and descriptive poems. She died in 1866, at the age of eighty-one years.

Edward was the fourth and youngest of their children. He graduated at Harvard College in 1841. His health became so much impaired during his college studies that he could not be with his class at Commencement, and consequently could not claim his place in the final distribution of honors; but, at the time when he left college, he stood first in rank. He had some question about the choice of a profession after graduating, but settled at last upon medicine, and went to Philadelphia to pursue his studies. After taking his medical degree in 1846, he travelled extensively in Europe with the eldest son of the late Mr. Abbott Lawrence. On returning to this country, he established himself as a physician in Boston. For some years, he made a specialty of diseases of the ear, in addition to his general practice. As the latter increased, he gave up the special branch, in which he was for a considerable period the principal, if not the only, expert generally recognized as such by our community.

His business increased rapidly, and included many of the leading families of the city and its neighborhood. He was often sent for to visit patients in distant places, and his consultations at home were resorted to by large numbers from all parts of the country.

Dr. Clarke united many, perhaps it would not be extravagant to say most, of the qualities which best fit a man for medical practice. His mind was at once inquiring, observant, reflective, and judicial. Some physicians are restlessly curious, but without the penetrating glance of the natural observer. Some are curious and penetrating, and pick up new and interesting facts in their foraging excursions, but never co-ordinate them by serious and continuous reflection. And some who are curious, observant, and reflective, are full of ingenious ideas, and it may be of useful suggestions and even sound opinions, yet want that decisiveness of character which makes its possessor choose his ground firmly after balancing evidence, where others waver and hesitate, and thus gives him that authoritative weight which is felt at every bedside and in every consultation.

Dr. Clarke had all these gifts, and added to them very great industry, entire concentration on his professional work, and that other requisite which commends all the rest to public favor, a manner and address eminently adapted to inspire confidence. It is not to be wondered at, therefore, that he obtained a great hold upon the respect and affections of a very wide circle of patients. In fact, his work became too much for his bodily strength, even before he was attacked with the first symptoms of his fatal malady.

This disease was a malignant affection of the lower portion of the intestinal tract, slow in its progress, distressing in its symptoms, inevitable in its consequences. If the highest test of philosophy or of Christianity is found in the manner in which the greatest trials of life are borne, it would be hard to find a better example of the practical illustration of either than was furnished by Dr. Clarke, during the three years of his slow martyrdom. He bore all his sufferings with wonderful patience and even cheerfulness. He kept himself busy with continuous thinking on a subject which had long interested him, and left a manuscript upon which he wrote so long as he was able to hold a pen, and until within a short time before his death. This manuscript, since published, shows no trace whatever, so far as I can see, of the pains as of a woman in travail in the midst of which it was written.

Although Dr. Clarke's life was chiefly devoted to medical practice, he found time for various other duties and offices. In 1855, he was chosen Professor of Materia Medica in the Medical School of Harvard University, which place he held until 1872, when he resigned.

As a teacher, he was singularly successful. He made a department commonly thought unattractive a favorite one with the students who listened to his lectures. He was considered, and deservedly so, a high authority on the subject to which his Professorship related, and wrote many articles for the new American Cyclopædia respecting various remedies.

His published works are the following: —

The Physiological and Therapeutical Action of the Bromide of Potassium and Bromide of Ammonium. By Edward H. Clarke, M.D., and Robert Amory, M.D. Boston, 1872.

Sex in Education; or, A Fair Chance for the Girls. Boston, 1873.

The Building of a Brain. Boston, 1874.

Visions: A Study of False Sight (Pseudopia). With an Introduction and Memorial Sketch by Oliver Wendell Holmes, M.D. Boston, 1878.

The last is the work referred to as having occupied much of his time during his illness; this has been published since his death.

In addition to his professional work and his published writings, Dr. Clarke took an active part in relation to various matters of general interest, especially the Public Park question and all subjects connected with the health of the city. Immediately after resigning his Professorship, he was chosen a member of the Board of Overseers of Harvard University, and shared in their deliberations up to a late period of his mortal illness.

This seems to be the well-filled record of a busy life of fifty-seven years. But, without knowing the great labor he spent upon the daily record of his cases, much would be passed by unreckoned. Some scores of large volumes containing these records have been burned, as he directed they should be, by his representatives since his death. To him they were of incalculable practical value, but the pathological biography of his fellow-citizens was never meant for public inspection.

JOHN LOTHROP MOTLEY.

John Lothrop Motley died in Dorchester, England, May 29, 1877.

He was born in Boston, April 15, 1814, and took his degree at Harvard College in 1831. He afterwards studied at the universities of Göttingen and Berlin, and on his return home went through a course of legal study, and was admitted to the Boston bar. In 1839, he published a novel, which made but slight impression, though it bore evidence of powers that were fully recognized at a later time. He was secretary of the American legation to Russia for a few months, and then came back to a life without much literary or other purpose, until about the year 1845, when he determined to write a History of the Netherlands, or more particularly of the revolt against Spain. "I had not first made up my mind," he afterwards wrote, "to write a history, and then cast about to take up a subject. My subject had taken me up, drawn me on, and absorbed me into itself. It was necessary for me, it seemed, to write the book I had been thinking much of, even if it were destined to fall dead from the press. . . . It was not that I cared about writing *a* history, but that I felt an inevitable impulse to write one particular history." After beginning the work, he found that he must seek its sources in Europe, and thither he went, not merely to travel, but to reside during much the greater part of his subsequent life. He made long and thorough investigations, especially at the Hague, Brussels, and Dresden; wrote with vigorous perseverance, and in 1856 brought out three large volumes on "The Rise of the Dutch Republic." "To all who speak the English language," he says in the preface, "the history of the great agony through which the Republic of Holland was ushered into life must have peculiar interest; for it is a portion of the Anglo-Saxon race, essentially the same, whether in Friesland, England, or Massachusetts. . . . The lessons of history and the fate of free States can never be sufficiently pondered," he adds, with more special reference to his own countrymen, "by those upon whom so large and heavy a responsibility for the maintenance of rational human freedom rests." The feeling for his subject, with which he began upon it, deepened and widened as he went forward; and, when he appeared as a historian, it was of movements affecting, as he had reason to think, the whole civilized world. We must appreciate this, if we would appreciate either the strong or the weak points of his work: on the one hand, its brilliancy, its fire, and its sweeping range; on the other, its want of balance or of penetration, its almost partisan character as it deals with those he passionately admired or as passionately abhorred. The success of the history was immense. It sold by thousands in the United States and in England, and was translated into Dutch, German, and French. All sorts of honors were bestowed upon the author, none greater than his election to the French Institute as Prescott's successor, in 1860. In that year, he published two volumes of the History of the United Netherlands, and in 1867 two more, completing the work. He closed it with these words: "The writer now takes an affectionate farewell of those who have followed him with an indulgent sympathy, as he has attempted to trace the origin and the eventful course of the Dutch Commonwealth. If by his labors a generous love has been fostered for that blessing, without which every thing that this earth can afford is worthless, — freedom of thought, of speech, and of life — his highest wish has been fulfilled." His most striking occasional production was one in entire harmony with the key-note of his histories. It was on the causes of the American Civil War, and appeared in the "London Times" in 1861. That same year, he was appointed United States

Minister to Austria, where he remained till 1867. In 1869, he became Minister to England, but held that position only a little more than a year. His third and last historical work was the "Life and Death of John of Barneveld," published in 1874. The three histories cover the three-quarters of a century, from about 1550 to about 1620, and from this point he intended to go on with the History of the Thirty Years' War. But his work was ended. Sorrow came, his strength failed; and, after a year or two of decline, he died peacefully. As a historian, he is distinguished alike for his original researches and for the striking use of them, with loftiness of sentiment, and an ardent devotion to great principles, but not for calmness or the judicial character which gives a history, in which it is prominent, the strongest assurance of a lasting place in literature. No tribute to him, however brief, should pass over his devotion to his country. A true American notwithstanding his long years in Europe, a true republican in presence of all older institutions to which his historic tastes would be naturally drawn, he changed his skies without changing his affections. The very last recollection of him, with the present writer, is the ready and even enthusiastic use he made of his great influence in Holland to procure a government publication for one of our libraries.

CHARLES PICKERING.

Charles Pickering, M.D., died in Boston, of pneumonia, on the 17th of March, 1878, in the seventy-third year of his age. He was of a noted New England stock, being a grandson of Colonel Timothy Pickering, a member of Washington's military family and of his first Cabinet as President; and he was elected into this Academy under the Presidency of his uncle, John Pickering. He was born on Starucca Creek, on the Upper Susquehanna, in the northern part of Pennsylvania, at a settlement made on a grant of land taken up by his grandfather, who then resided there. His father, Timothy Pickering, Jr., died at the age of thirty years, leaving to the care of the mother — who lived to a good old age — the two sons, Charles and his brother Edward, who were much united in their earlier and later lives, and were not long divided in death, the subject of this notice having been for only a year the survivor.

Dr. Pickering was a member of the class of 1823 at Harvard College, but left before graduation. He studied medicine, and took the degree of M.D. at the Harvard Medical School in 1826. Living in these earlier years at Salem, he was associated with the late William

Oakes in botanical exploration; and it is believed that the two first explored the White Mountains together, following in the steps of the first botanist to ascend Mount Washington, Dr. Manasseh Cutler of Essex County, and of Francis Boott and the still surviving Dr. Bigelow. His taste for natural history showed itself in boyhood, both for botany and zoölogy, and probably decided his choice of a profession. He may have intended to practise medicine for a livelihood when, about the year 1829, he took up his residence at Philadelphia; but it is probable that he was attracted thither more by the facilities that city offered for the pursuit of natural history than by its renown as a centre of medical education. We soon find him acting as one of the curators of the Academy of Natural Sciences, and also as librarian, and with reputation established as the most erudite and sharp-sighted of all the young naturalists of that region. His knowledge then, as in mature years, was encyclopædic and minute; and his bent was toward a certain subtlety and exhaustiveness of investigation, which is characteristic of his later writings. Still, in those days in which he was looked up to as an oracle, and consulted as a dictionary by his co-workers, he had published nothing which can now be recalled, except a brief essay on the geographical distribution and leading characteristics of the United States flora, which very few of our day have ever seen.

When the United States surveying and exploring expedition to the South Seas, which sailed under the command of then Lieutenant Charles Wilkes in the autumn of 1838, was first organized under Commodore T. Ap-Catesby Jones, about two years before, Dr. Pickering's reputation was such that he was at once selected as the principal zoölogist. Subsequently, as the plan expanded, others were added. Yet the scientific fame of that expedition most largely rests upon the collections and the work of Dr. Pickering and his surviving associate Professor Dana, the latter taking, in addition to the geology, the Corals and the Crustacea, and other special departments of zoölogy being otherwise provided for by the accession of Mr. Couthouy and Mr. Peale. Dr. Pickering, although retaining the ichthyology, particularly turned his attention during the three and a half years' voyage of circumnavigation to anthropology, and to the study of the geographical distribution of animals and plants; to the latter especially as affected by or as evidence of the operations, movements, and diffusion of the races of man. To these the subjects of his predilection, and to investigations bearing upon them, all his remaining life was assiduously devoted. The South Pacific exploring expedition visited very various parts of the world; but it necessarily left out regions of the highest interest to the anthropological investigator, those occupied in early times by the race to which we belong, and by the peoples with which the Aryan race has been most in contact. Desirous to extend his personal observations as far as possible, Dr. Pickering, a year after the return of the expedition, and at his own charges, crossed the Atlantic, visited Egypt, Arabia, the eastern part of Africa, and western and northern India. Then, in 1848, he published his volume on "The Races of Man, and their Geographical Distribution," being the ninth volume of the Reports of the Wilkes' Exploring Expedition. Some time afterwards, he prepared, for the fifteenth volume of this series, an extensive work on the Geographical Distribution of Animals and Plants. But, in the course of the printing, the appropriations by Congress intermitted or ceased, and the publication of the results of this celebrated expedition was suspended. Publication it could hardly be called: for Congress printed only one hundred copies, in a sumptuous form, for presentation to States and foreign courts; and then the several authors were allowed to use the types and copperplates for printing as many copies as they required and could pay for. Under this privilege, Dr. Pickering brought out in 1854 a small edition of the first part of his essay, — perhaps the most important part, — and in 1876 a more bulky portion, "On Plants and Animals in their Wild State," which is largely a transcript of the note-book memoranda as jotted down at the time of observation or collection.

These are all his publications, excepting some short communications to scientific journals and the proceedings of learned societies to which he belonged. But he is known to have been long and laboriously engaged upon a work for which, under his exhaustive treatment, a lifetime seems hardly sufficient; a digest, in fact, of all that is known of all the animals and plants with which civilized man has had to do from the earliest period traceable by records. When Dr. Pickering died, he was carrying this work through the press at his own individual expense, had already in type five or six hundred quarto pages, and it is understood that the remainder, of about equal extent, is ready for the printer. This formidable treatise is entitled "Man's Record of his own Existence." Its character is indicated in the brief introductory sentences: —

"In the distribution of species over the globe, the order of Nature has been obscured through the interference of man. He has transported animals and plants to countries where they were previously unknown; extirpating the forest and cultivating the soil, until at length the face of the globe itself is changed. To ascertain the amount of this interference, displaced species must be distinguished, and traced each to its original home. Detached observations have already been given in the twenty-first and succeeding chapters of my 'Races of Man;' but, when such observations are extended to all parts of the globe, the accumulated facts require some plan of arrangement. A list will naturally assume the chronological order, beginning with Egypt, the country that contains the earliest records of the human family, and receding geographically from the same central point of reference."

Then, starting with "4713 B.C.," and "4491 B.C., beginning of the first Great Year in the Egyptian reckoning," he begins the list, which, under the running heading of "Chronological Arrangement of Accompanying Animals and Plants," first treats of the vegetables and animals mentioned in the book of Genesis, and of the "Commencement of Bedouin or Nomadic Life in the Desert;" passes to the "Colonization of Egypt," and to critical notices (philological and natural-historical) of its plants and animals, as well their earliest mention as their latest known migrations; reaches the beginning of the Christian era at about the 470th page; and so proceeds, till our wonder at the patience and the erudition of the writer passes all bounds. We are ready to agree with a biographer who declares that our associate was "a living encyclopædia of knowledge," — that there never was a naturalist "who had made more extended and minute original explorations;" and we fully agree that "no one ever had less a passion or a gift for display;" "that he was engaged during a long life in the profoundest studies, asking neither fame nor money, nor any other reward, but simply the privilege of gaining knowledge and of storing it up in convenient forms for the service of others;" that "the love of knowledge was the one passion of his life," and that "he asked no richer satisfaction than to search for it as for hidden treasure." He was singularly retiring and reticent, very dry in ordinary intercourse, but never cynical; delicate and keen in perception and judgment; just, upright, and exemplary in every relation; and to those who knew him well communicative, sympathetic, and even genial. In the voyage of circumnavigation, he was the soul of industry, and a hardy explorer. The published narrative of the commander shows that he took a part in every fatiguing excursion or perilous ascent. Perhaps the most singular peril (recorded in the narrative) was that in which this light-framed man once found himself on the Peruvian Andes, when he was swooped upon by a condor, evidently minded to carry off the naturalist who was contemplating the magnificent ornithological specimen.

Dr. Pickering married in the year 1851, and leaves a widow, but no children to inherit this honored name.

EDMUND QUINCY.

Just on the verge of the ninety-eighth year of the Academy, the painful intelligence was received of the sudden decease of one of its most esteemed members, who had served for several years as its Treasurer. Mr. Edmund Quincy descended from a family among the earliest to leave Great Britain for the purpose of settling upon the soil of Massachusetts, and which has actually fulfilled that object on to three centuries, continuously. The second son of Josiah Quincy and the grandson of Josiah Quincy, known as the *Junior*, both of them doing honor to the name under high political responsibilities, Edmund did not fail to maintain their reputation, though not precisely in the same way. When nine years of age, he was sent to Phillips Academy at Andover, for preparation to enter Harvard College; and in 1827 he issued from that institution with honors indicating a fair promise of distinction in his later years. That promise was honorably fulfilled.

At the outset of life, the usual question presents itself to educated men in New England, what of three professions they decide to take. Mr. Quincy preferred the law; but, though he went through the preliminary preparations, he developed less taste for it than for the cultivation of general literature and the occupation of a writer. Hence it happened that through an elaborate experience he gradually mastered a style of composition marked not less for its peculiar felicity than for its accuracy and point.

Mr. Quincy married early, and then settled himself in one of the ancestral mansions in Dedham, which had come in due course of inheritance to the possession of his father. For a short period, it looked as if there might be danger of his subsiding into the respectable but somnolent career of a fastidious critic about town. His early effort naturally could hardly be more than ephemeral productions which get mingled with more or less of the platitudes that shine for a moment and forthwith are seen no more. In order fully to draw out his vigor, there was need of some strong appliance in the living and acting world around him. Just the thing happened in its most striking aspect, when in the month of November, 1837, there came the ghastly intelligence from the town of Alton, in the State of Illinois, that a respectable clergyman had there been deliberately murdered by a ruthless mob solely on account of his persisting to substitute a second printing-press, with the purpose of exposing the wrongfulness of negro slavery, for an earlier one which had been ruthlessly destroyed. Perhaps no single event in the history of the long struggle that followed stirred up conscientious men to a sterner sense of the necessity of exertion than that event. It roused Mr. Quincy at once, and from that date he stood forth an altered being. He had found a work to do, and he faithfully performed it.

But, startling as this intelligence appeared and incontestably was, so thoroughly had the popular mind in the good city of Boston been imbued with a dread of the possible consequence of agitating the question of slavery in any shape, that great sluggishness, to use a mild term, was felt towards any public condemnation of the true nature of that crime. At this day, it would not be easy for young generations to conceive of the extent of the popular prejudice on this subject. No doubt, it sprang from an honest apprehension of the consequences to the much loved Federal Union which might even bring on its disruption. Such was the feeling almost all over the land. And nowhere was it more overpowering than in the city of Boston. Yet in the midst of the excitement there appeared a few brave individuals, men and women, who, being shocked at the idea of suffering this wanton outrage to pass without publicly stamping upon it their sense of its nature at any cost, assembled for consultation, and these finally agreed upon a public call to all persons sympathizing with them to meet tegether and deliberate upon what might be done to stigmatize the true nature of the offence.

This meeting was accordingly held. And of those persons regardless of consequences to themselves, but strongly moved by the atrocity of the outrage upon freedom, Edmund Quincy appeared as one. If on this issue there was to be a conflict of principle, his mind was altogether made up. It was now that he conceived the idea that there was no alternative but to enlist actively for the whole of the war, be it longer or shorter. His speech made on that night was the key of his career.

What a change came over the person of Mr. Quincy by reason of the bold step he had taken can be understood only from an examination of the papers he has left behind him. Those who sympathized with him were a handful. Utterly unsuited to the arts of a demagogue, it became at once his task to attack with severity almost the whole of the class of persons of property and of standing, of all the higher professions, and of advanced culture, naturally his associates, who stood forth almost in a body to protect what they honestly believed was threatened with destruction, the union of the States and the property of the nation. With very little sympathy for the style of electioneering so common in the country, detracting from the rich and exalting the poor man for no other reason than the fact itself of either position, it became his work not to spare the numerous class of those who make these labors their sole occupation. So long as the slave remained in chains, the demagogues were mostly arrayed on the side of the masters. It was this class that it became the business of Mr. Quincy to assault, and he did not spare them. How much work he did as a regular contributor to the chief anti-slavery presses for a long series of years can be understood only from the collection from them made by himself and enclosed in a series of ponderous volumes. It is here, and perhaps here only, that a very full political exposition of that struggle may yet be collected. It makes a memorable history, second only to that of the war for independence. The most valuable feature of it is its freedom from personal or party motive. Mr. Quincy never sought an office or peddled for a place. In a word, he was thoroughly independent, a quality more often praised than practised among men of his class, when they undertake to meddle with politics at all.

Neither was it only in the field of controversy that he exercised his pen. It was early in his career that he ventured upon a work of fancy. This was a small volume issued under the name of "WENSLEY, A TALE;" and the scene of action purported to be laid in New England somewhere about the middle of the last century. It aimed to represent neither the more polished nor the purely homely phases of life, all which had been shown well enough already elsewhere, but rather the quiet and measured retirement of the middling but educated class settled in districts rather remote from the populous seaboard, and yet not wholly out of reach. The story is simply developed through the agency of four or five characters of both sexes, and the happy union of the hero and heroine in spite of the wicked contrivances of an English rival to defeat it. As an experiment, it was certainly not without interest. Its greatest recommendation consists in its easy vein of humor, of a sort much removed from that which under the name of *Yankee* has been carried too near the extreme of vulgarity of late years. The characters may not appear to excite much sympathy under their trials, but at least they are well sketched, and the dialogue retains salt enough everywhere to hold the attention and leave at the end good-will.

This book of "Wensley" and another of the same sort, which he prepared for a magazine, but did not publish, formed the recreation of Mr. Quincy. His more elaborate work is to be found in the continuous history, furnished by him to the anti-slavery presses, of the fearful political struggle for the extermination of slavery, in which he took so prominent a part. Nowhere else is it so minutely and faithfully told. In addition to this, he was regularly called upon to furnish reports of proceedings of public meetings, addresses, speeches, and, last but not least, a share in the production of those New Year's Annuals, so tastefully prepared both inside and out, and made attractive not less to the mind than to the outward eye, which contributed in no small degree to keep up a general interest in the great cause and to hasten its final triumph.

That hour came at last: the great object of emancipation was attained after a conflict of nearly half a century, and not without a fearful penalty of bloodshed. Little remained for him beyond the comparatively light labor of securing results under the strongest possible cohesion of liberty with law. Mr. Quincy, had he so chosen, might justly claim a complete release from controversy of every kind, and especially from the ever-recurring demands of political newspapers. He did, in fact, turn his hand in a different direction, and, instead of laboring to establish present or future history, he directed his attention to the illustration of the past.

One special duty rested upon him, to perform which was a task nobody else could do so well. His father, Josiah Quincy, and John Copley, two of the most eminent men of their age, were born in the same town of Boston so nearly together as to have been nursed by the same nurse; and, though soon widely separated by the Atlantic in their respective paths of usefulness, they had the singular fortune of each extending a useful and honorable career close upon an entire century. While Quincy labored through many of the various stages of active life, in the representative halls, on the bench of justice, in the organization of his native place as a city, and lastly in the faithful supervision of one of the first universities in the land, it fell to the lot of his rival to rise by regular degrees through all the various stages of distinction that attend an eminent parliamentary orator until elevated to the highest judicial position which Great Britain could give. A singular feature of this conjunction was that these two persons not only should have been so close together at birth, but that they should have continued their laborious and useful lives almost to the same day of termination. Mr. Quincy was the survivor but a few weeks. It was no more than an act of justice in his son to show to the public an example of so long, so industrious, and so useful a career. It scarcely needs to be added that it was so judiciously done as to secure for it shortly a second edition, an event seldom attending any production of that sort not possessing intrinsic worth.

Neither was this the last of his labors. His interest in the researches in which he had been so zealously engaged led him to collect and prepare for publication a volume of the speeches made by his father during the period of his active life. This was likewise well received by the public. Here his labors ended. His observation of the progress of the instruction, when elected as one of the Trustees of Harvard University, was earnest; and it led him to act as an occasional visitor to listen to the exercises of the students. It was after a visit of this kind that on the 17th of April, 1877, just as he got home to his own doorstep at Dedham, the fatal stroke fell to terminate in an instant his most industrious and honorable career.

JOHN H. TEMPLE.

Mr. John H. Temple was born in Princeton, Mass., on Oct. 3, 1812. He died in West Roxbury on July 25, 1877. His parents were farmers, and were healthy and vigorous even to old age. The son was of a delicate and sensitive nature. His whole life was a struggle with a nervous and frail constitution, and in his mature years he suffered from asthma. He left the paternal farm at eighteen for Sterling, where he was employed in the manufacture of chairs. At twenty, he began to work on physical apparatus under the instruction of Mr. Nathan B. Chamberlain. He came with him to Boston, and remained in his service for several years; after which, he began business for himself, about 1838. For fourteen years, his humble shop was in Court Street; he then removed to Franklin Street, and about 1865 to West Roxbury. At first, he manufactured apparatus of illustration for schools and colleges, and for the Lowell Institute in its early days. But his taste was always inclined to mathematical instruments and instruments of precision, in the construction of which he excelled, and to which he devoted all the energies of the best part of his life. The officers of the United States Coast Survey, and engineers generally, appreciated his skill and his conscientious fidelity to a high ideal of workmanship, and engrossed all his time. His standard of execution was so high, and he found it so difficult to satisfy himself even with the results of his own labor, that he could rarely obtain any valuable assistance at the hands of others. Under such circumstances, his business was highly honorable, but not remunerative. Theoretical and practical science must ever acknowledge their obligation to the genius of the workshop, whose inventive faculty and nice instrumental appliances make the discoveries of the laboratory possible. It is fitting that services which are poorly paid in coin should receive their due share of honor. So thought the members of the Academy when they elected Mr. Temple a Fellow in 1845; the first of his class to enjoy a distinction in which only two others have since shared. He was a man to whom this unsought honor was more than money, and he clung to his membership, at some sacrifice which he could ill afford, for thirty-two years. He probably never attended a meeting, certainly not more than one or two; partly on account of his excessive modesty and self-depreciation, but partly, no doubt, because of an absorbing occupation, too great for his physical strength.

Nothing characterizes the science of the present day so much as its aspiration for nicety of measurement in time and space; and nothing limits the flights of its ever expanding wings but the unavoidable errors of workmanship in the instruments it employs. The crowning work of Mr. Temple's life was the conception and construction of a dividing-engine, which takes rank of all other instruments because it is the instrument by which instruments themselves are made. He had not seen a dividing-engine when he began the construction of his own in 1852, and it is believed that he never saw any one but that which he lived to complete. All his hours of leisure, all the money which he could spare from his frugal style of living, and many sleepless nights for twenty years, was the price which he ungrudgingly paid for the object of his ambition. But he finished his work, and in time to use it in the manufacture of his own instruments. The conception and the execution of the dividing-engine were the undivided product of his brain and hands. Strong as his own will, but delicate as his own fine organization, it was his pride in life, and is now his monument. One hundred years before Mr. Temple began to build it, Ramsden, in England, had made the first dividing-engine, and Troughton, who was to win new victories in mechanical skill, had just opened his eyes to the light of day. But the fame of both, and also of their worthy compeer in France, Gambey, still survives in the veteran instruments which adorn the observatories of Europe, and divide with the astronomers the triumphs of discovery.

Competent judges have pronounced the dividing-engine of Mr. Temple at least equal, in solidity and delicacy, to the best in the world. In his own line of work, he had no superior, perhaps not an equal in this country. And he created the standard of excellence which he then tried to attain. With so much of which he might justly boast, he was always oppressed by a sense of his own shortcomings, and he required the encouraging word of friends to make him just to himself.

And this encouragement did not fail him. For his sweet and attractive countenance, his modest demeanor, his gentle nature, and a native refinement which art can but poorly imitate, enlisted the good-will of all with whom he was associated. Science is the gainer when she claims him as one of her own children.

JOHN E. TYLER.

We have to record the death of still another of our associates by that disease which has of late proved so fatal to professional and scientific men. Dr. Tyler died with pneumonia on the 9th of March last, after a very brief illness. He was born in Boston, Dec. 9, 1819, and was the second son of John E. and Hannah Parkman Tyler, of Westborough, Mass. His father, a graduate of Harvard in 1786, was educated a physician, but afterwards became engaged in business in Boston. Dr. Tyler was himself early destined to a mercantile life, and developed an aptitude for business which was of much service to him in the executive offices he was called to fill in later life. His preliminary education was begun in Westborough, and continued in Leicester and Phillips (Andover) Academies. He entered the Freshman Class of Dartmouth College in 1838, and graduated in due course and with high honors in 1842. Here Tyler gave evidence of that ready wit and humor which was always a conspicuous element in his nature, and which, added to brilliant scholarship, gave him an immense popularity in his class. He was foremost in all athletic games and sports. He was a fine musician, a singer, and an adept upon several instruments. He was also a good writer and an easy and graceful speaker. He was a member of the Phi Beta Kappa and Psi Upsilon Chapters, and president of the United Fraternity, one of the two leading literary societies of the college.

Almost immediately after his graduation from college, he went to Newport, R. I., where he entered upon the study of his chosen profession under the guidance of the late Dr. Dunn of that city. He subsequently attended a course of medical lectures at Hanover, and two sessions at the medical department of the University of Penn, in Philadelphia, at which latter institution he graduated in the spring of 1846. He also received a medical diploma at Hanover.

Dr. Tyler first entered upon the practice of his profession at Salmon Falls in New Hampshire. While there, he was sent to the State Legislature, and was soon called to take charge of the New Hampshire Asylum for the Insane at Concord, where he remained till he was appointed to the honored post of physician and superintendent of the McLean Asylum for the Insane at Somerville, made vacant by the resignation of Dr. Bell. This was in 1858. Here he remained till the spring of 1871, when he was compelled by failing health to offer his resignation. It was during this long term of service at Somerville that Dr. Tyler showed that marked executive ability, sound judgment, knowledge, and skill which have made his name famous in this and in other countries. His official reports while at the head of the McLean Asylum have been largely quoted, and are recognized by the profession as among the ablest and best in this department of medical literature.

Dr. Tyler twice visited Europe, where he enlarged and enriched his knowledge of his favorite science, and was received by his *confrères* in the Psychological Associations of Great Britain and Ireland with marked courtesy and attention. Upon his retirement from hospital life, he took up his residence in Boston, where he soon acquired a large consulting practice in his specialty. In 1871, he was appointed to the chair of mental diseases in the medical department of Harvard University, having previously been connected with the Medical School as University lecturer on the same subject. In recent years, Dr. Tyler held several important posts in connection with our city and State commissions. He was also a trustee, under the will of the late Seth Adams, of the proposed institution for the treatment of nervous diseases. In all these official capacities, as well as in his professional and social relations, Dr. Tyler was a man of singularly pure and unblemished life. He was a devoted and successful physician, an exact scientist, a faithful and conscientious worker in the difficult and delicate sphere of duty in which for the greater part of his life he was especially called to serve.

J. P. KIRTLAND.

Dr. J. P. Kirtland died in Cleveland, Ohio, Dec. 10, 1877, aged eighty-five years. He was one of the last of our older naturalists like Say, Audubon, and Henry, — men who were young when zoölogy and physics were young, and who, from an inborn love of nature and an enthusiasm for knowledge, were enabled to create methods and to make discoveries. He was born in Connecticut, and even in boyhood showed a strong taste for horticulture, so that at twelve he had a neat garden of his own, and was a skilful budder and grafter. He studied too the Linnæan system of botany, raised silk-worms, and began bee culture, a pursuit he steadily continued for nearly seventy years. After studying medicine at Yale College and in Philadelphia, and after some years of practice in Connecticut, he moved to Ohio, where he spent the remainder of his long life. While busily following his calling of physician, he found time for a great deal of other work. During a quarter of a century, he was professor of medicine. In 1848, he worked up the natural history of Ohio, as part of the geological survey of that State. Not the least valuable portion was an account of the fishes, which was published, with plates, in the "Journal of the Boston Society of Natural History," and which still stands as a work of authority. He wrote also valuable papers on sexualism among the naiades. The growing of fruit he pursued during his whole life, and was very successful, especially in producing new varieties of cherries. It is scarcely necessary to add that in this respect he was a public benefactor.

Such a man is always interesting. The peculiarities which make him what he is, and the native energy and originality which have held him up, give a certain freshness of character rarely found among men of strictly academic training.

ELIAS MAGNUS FRIES.

Elias Magnus Fries died at Upsal on February 8, in the eighty-fourth year of his age, five months after the celebration, in which he was able to take some part, of the four hundredth anniversary of the foundation of that University, and a month after the hundredth anniversary of the death of Linnæus. Born, as was Linnæus, in Smoland, a southern province of Sweden, and like him called in middle age to the renowned Scandinavian University, he might be regarded as the most distinguished of Linnæus's successors, except for the fact that he did not occupy the chair of Linnæus; for when, more than forty years ago, Fries, then Demonstrator of Botany at Lund, was called to Upsal, Wahlenberg was in the botanical chair, and Fries was made professor of Practical Economy. His son, however, by the retirement of Areschoug, is now the botanical professor.

Fries's earliest work, the first part of his Novitiæ, appeared in the year 1814, when the author was only twenty years old. His last of any moment, a new edition of his Hymenomycetes Europæi, was published on his eighty-first birthday, Aug. 15, 1874. Most of the sixty intervening years are marked by some publication from his busy and careful hand. His work was wholly in systematic botany, and of the highest character of its kind. In phænogamous botany, it related chiefly to the Scandinavian flora, in which for critical judgment he had no superior; in Mycology, of which he was the reformator, and to a good degree in Lichenology, he had no rival except as regards microscopical research. The modern microscope did not exist when he began his work, and, while showing how much can be done without it, he may too long have underrated its value. But he lived to see it confirm many conclusions which his insight foresaw, and solve riddles which he had pondered, but was unable to divine. He was the prince, Nestor, and last survivor of an excellent school of systematic botanists, whose teachers were taught by Linnæus or his contemporaries.

URBAIN-JEAN-JOSEPH LEVERRIER.

URBAIN-JEAN-JOSEPH LEVERRIER was born at St. Lô, in the department of the Manche, on March 11th, 1811. As a boy, he studied at the colleges of St. Lô at Caen, and in Paris at the College of Louis le Grand. In 1831, he entered the École Polytechnique, where he graduated with such distinction that he was allowed to choose which branch of the public service he would enter. Obtaining a position in the tobacco bureau, he devoted his leisure to chemistry, and published, as his first contribution to science, two papers on the combinations of phosphorus with hydrogen and oxygen. His natural tastes, however, were in the direction of the mathematics, and soon after, receiving a minor appointment in the École Polytechnique, he was enabled to devote his entire energies to his favorite science. At the instigation of Arago, he undertook the examination of the mutual disturbances of the planets, a subject to which he devoted a large portion of his life. A complete discussion of the motion of a single planet is a work of which any astronomer might be proud, but the determination of the motions, and the formation of tables for computing the positions of all the planets is a work of such magnitude that it would seem beyond the powers of a single individual. Yet LeVerrier not only boldly undertook this problem, but carried it to a successful termination, and built himself a lasting monument in the superb volumes of the "Paris Observatory," in which these researches are published.

The discovery of Neptune, by which LeVerrier is best known to the public, enters as a small portion of this great work. A study of the discordance in the motion of the planet Uranus from its path, as given by theory, led him to suspect the existence of an outer planet, producing the disturbance by its attraction. An investigation of the mass and position of such a body enabled him to present to the Academy, on June 1, 1846, a paper predicting the position of the unknown planet. Three months later, Galle examined this portion of the heavens at the request of LeVerrier, and discovered a star within two degrees of the computed place, which was not on the maps, and which proved to be the new planet. This discovery was at once received with the greatest enthusiasm. Honors poured in on LeVerrier from every side, and it was even proposed to name the planet from him. Fortunately, however, as in the case of Uranus, cooler judgment prevailed, and the precedent of naming the planets from the Roman deities was not broken. It afterwards appeared that the English mathematician, Adams, was engaged on this same problem, though by a less rigorous method, and an equal share of the glory of the discovery was claimed by his friends for him. It has also been shown that by making different assumptions LeVerrier might have arrived at widely different results. The fact, nevertheless, remains, that LeVerrier was the first to predict on theoretical grounds the true position of the unknown planet, and that in consequence of this prediction Neptune was discovered.

On the death of Arago in 1854, LeVerrier was appointed his successor as Director of the Paris Observatory. In 1870, he was removed from this position, but reinstated in 1873, when he resumed the publications of the "Annals of the Observatory," which had been discontinued during his absence. Although a mathematician rather than an observer, he introduced many important changes in the work of the Paris Observatory, and greatly increased its efficiency. For many years, he was a senator and member of the Superior Council of Public Instruction, and was thus enabled to render material aid to the cause of higher education in France. He was the originator and President of l'Association Scientifique de France, and to him is its success largely due. During the last year of his life, he was much interested in International Meteorology, and succeeded in establishing a great number of stations in France. After an illness of about six months, he died on the morning of September 23d, 1877, on the thirty-first anniversary of the discovery of Neptune.

HENRI VICTOR REGNAULT.

Henri Victor Regnault was born at Aix-la-Chapelle on the 21st of July, 1811, and died in Paris on the 19th of January of the present year. He obtained, while still a lad, a position in a drapery establishment in Paris; but after some time was able to enter the École Polytechnique, where he remained two years. After spending eight years in the Department of Mines, he obtained a professorship at Lyons, and entered upon the field of research of organic chemistry. The peculiar character of his mind showed itself at once in his new career. He paid no attention to the theories of the day, but worked diligently at the accumulation of materials. A great number of valuable investigations soon followed. Among them, we may notice especially his researches on the action of chlorine upon ether, and upon the chlorides of ethyl and of ethylene; researches which still retain their value and interest, as the physical properties of the bodies which he obtained were studied with unusual care and thoroughness. In 1840, Regnault was appointed professor in the École Polytechnique, and in 1841 he became Professor of Physics at the Collége de France. There he began his life-work in physics by a careful and masterly study of the specific heats, first of the elements, and afterward of compounds. He devised for this study the calorimeter which bears his name, and his results have, to the present day, been standards of accuracy. He established the law of Dulong and Petit for the greater number of the elements, and showed that for many compounds the atomic heat of the whole is the sum of the atomic heat of its constituents. He next undertook, by order of the Minister of Public Works, a series of investigations to determine the principal laws and numerical data which are required in the theory of the steam-engine. Then followed the finest series of experimental determinations of physical constants which has ever been executed by one man, in any age or of any nation. Ten of these memoirs are contained in Volume XXI. of the Memoirs of the Academy of Sciences in Paris, and three others of great length in Volume XXVI. of the same work. These papers, now familiar to all physicists, embrace the following subjects: —

1. The Expansion of Gases and Dry Vapors, the Coefficients being determined under a Constant Pressure and under a Constant Volume; at High and at Low Pressures.
2. The Determination of the Densities of Gases.
3. The Determination of the Weight of a Litre of Air, and of Nitrogen, Oxygen, Hydrogen, and Carbonic Dioxide.
4. On the Measure of Temperatures.
5. On the Absolute Dilatation of Mercury.
6. On the Law of the Compressibility of Elastic Fluids.
7. On the Compressibility of Liquids, and especially that of Mercury.

8. On the Elastic Forces of the Vapor of Water at Different Temperatures.
9. On the Latent Heat of Aqueous Vapor at Saturation under Different Pressures.
10. On the Specific Heat of Water at Different Temperatures.
11. On the Specific Heat of Elastic Fluids.
12. On the Elastic Forces of Vapors.
13. On the Latent Heats of Vapors under Different Pressures.

For every one of these investigations, an original method was pursued, and original apparatus was devised. The numerical results obtained form the basis of the modern science of thermics, and are quoted upon almost every page of works on the higher generalization known as thermo-dynamics. The memoirs cited above are by no means, however, the only contributions which Regnault made to his favorite branch of physics. A great number of minor papers contain important additions to our knowledge of physical data, or to our instrumental means of research. From time to time, he resumed and added to the work of his earlier years, taking up single and special points for investigation. In 1847, Regnault published a work on chemistry in four volumes, written with remarkable clearness, and containing many physico-chemical methods which are still in use; as, for example, a very elegant exposition of the theory and use of two of his own forms of the air-thermometer. This work was translated into several languages, and passed through several editions. In 1854, he became director of the porcelain manufactory at Sévres. During the war with Germany in 1870, Regnault lost his son Henri, an artist of extraordinary promise; and, after the final treaty of peace, he returned to his laboratory to find that the results of an elaborate investigation on the heat of expansion of gases had been completely destroyed during the German occupation of the town.

Regnault possessed in a remarkable degree the talent for devising apparatus and methods for the determination of physical constants. It is safe to say that with him began a new era in experimental physics. His mathematical powers were at least respectable, yet he seems never to have employed the modern mathematical processes for the treatment of his numerical results. He never devised experiments which, like sounding-lines, reached the depths of the unknown. Experiment was not with him, as with Faraday, an instrument of discovery, but only a most refined and beautiful instrument of observation. He never theorized, he drew no deductions from his own work, but he laid at the feet of the great architects of science grand and shapely blocks of material, with which they built and are still building. He seems to have had no conception whatever of the modern science of Energy or even of the principle of equivalent transformations, and yet this whole branch of knowledge has grown up since he began to work, and he himself largely, though indirectly, contributed to its growth. Let us not undervalue his rare and beautiful talent, — a talent which rose almost to the level of genius. For, if there are higher qualities of intellect, there are none which are upon the whole more useful, or which contribute more to the advancement of physical science.

LOUIS ADOLPHE THIERS.

Louis Adolphe Thiers, the veteran Statesman and Historian of France, died near Paris on the 3d of September last, in the eighty-first year of his age. He was born at Marseilles, on the 16th of April, 1797. Without any early advantages of family or fortune, he won for himself a name and a fame which will not soon be forgotten. He was a man of untiring industry, of extraordinary intellectual vigor, and of intense ambition. Distinguishing himself first as a Journalist in Paris, he soon turned his pen to the preparation of a History of the French Revolution from 1789 to 1799, and had published ten volumes before he had reached his thirtieth year. After an interval of twenty years, he resumed his historical labors; and, between 1845 and 1857, sixteen or seventeen volumes of his great work, "L'Histoire du Consulat et de l'Empire," were given to the press. Meantime, he had been a leading and devoted member of the Chamber of Deputies, and more than once a Minister of State, under Louis Philippe. But his most important political services were rendered after the fall of the Second Empire. His negotiations with Bismarck, and his liberation of the territory of France from foreign occupation, were conducted in a manner, and with a success, which commanded the admiration of his whole country; and he was soon hailed, almost by acclamation, as the First President of the new Republic. He had resigned that office before his death; but the Republicans of France still looked to him as their ablest and most skilful counsellor, and relied on him in every hour of difficulty and danger. He maintained to the last that the Republic was the only form of government then possible for his country, and never ceased to urge upon the people to show that "the Republic is a government of order, peace, and liberty." While Thiers, at the period of his death, thus stood foremost among the statesmen of France, he held also no second rank as a writer and an author;

and his name, in the order of date and of merit, was at the head of the roll of the French Institute. He was elected a member of this Academy in 1874.

COUNT PAUL FREDERICK SCLOPIS DE SALERANO.

This distinguished statesman was born at Turin in the year 1798, and after a long career of public service ended his life at the mature age of eighty years.

His education had been studiously cared for by his father, and he issued from all the courses prescribed in his natal city with distinguished honors. Neither was it long before he received an appointment in the department of the Minister of the Interior. From this point, his assent was easy to the judicial department, and to the Senate of Piedmont, then constituting the Superior Court of the nation. From this he was advanced to the still higher position of chief of the domestic service, and official counsel of the crown in matters of law. In the year 1837, he was selected as one of the commission to codify the Civil Code of Sardinia; and ten years later he was made President of the highest board of revision in the kingdom.

The events of the great year 1848, which went so far to shake all established forms of government in Europe, only contributed to mark Count Sclopis the more as a prominent statesman. Much against his will, he was compelled to assume the high post of keeper of the seals, as well as minister both of justice and of ecclesiastical affairs. He was likewise made President of a Commission to which was intrusted the duty of supervising the law touching the freedom of the press, his reports upon which have been recognized to this day as the most liberal in Europe. In the general election which followed, he was chosen a deputy from Turin. At this time, he carried through two of the most critical measures of that period. The first was a general amnesty necessary to restore quiet to the elements distracted by so much civil commotion. The second was a not less important provision for securing the liberty of the press. A year or two later, he was called in 1850 to the Senate, and at once elevated to the presidency of that distinguished assembly.

Having passed a great part of his life crowned with so many honors, when the day came that the course of events so far enlarged the territorial limits of the kingdom as to impose on him the necessity of transferring himself to a new capital, Florence, and ultimately at Rome, he could not reconcile himself to leave, in his old age, the place of his birth, and so he respectfully asked leave to retire to private life.

Thus it was that Count Sclopis remained in voluntary retirement at Turin, but it was not to waste his time in idleness. He had always been a voluminous writer, and he still continued his labors. One volume of the "History of Piedmontese Legislation," three volumes on "Italian Legislation," and several disquisitions on the "Political History of Savoy," at once showed the continued activity of his mind as well as the value of his investigations.

Although this decision of Count Sclopis necessarily threw him for a time into private life, it was not in the nature of things that the sovereign could keep him out of his mind altogether. In due course of time, an event occurred of a wholly novel nature in the history of the world. Two great nations which had what they considered as complaints to make of each other, instead of going to war and doing as much reciprocal injury as possible, agreed upon a mode of arriving at a settlement without fighting. This was in the form of a treaty, which provided for the construction of a board of arbitrators, representatives of their respective nations, whose province it should be to consider the arguments presented on their behalf, and to return an award understood to be conclusive on both the contestants. Such was the tribunal composed of representatives selected by the authorities of three entirely neutral nations, who, in conjunction with one from each of the aggrieved parties, should assemble at Geneva to hear and decide upon the merits of the questions as presented to them by their respective servants learned in international law. Such was the tribunal, well known as the arbitration held at Geneva in Switzerland in the year 1872. The three nations solicited to send arbitrators on this occasion were Italy, Brazil, and Switzerland; and they, in conjunction with a similar representative from each side, constituted the board of final appeal.

On behalf of the Kingdom of Italy, the sovereign, not unmindful of the ample qualifications of his ancient adviser, pitched upon Count Sclopis as his representative. And when the time came round for the assembling of the arbitrators at Geneva, and they met in council, there was not a moment's hesitation in selecting that learned and distinguished individual to preside over the deliberations. It is needless to add that he performed that duty with a moderation and a dignity entirely in keeping with the magnitude of the occasion. He had then reached the advanced age of seventy-four, a period when it might fairly be permitted to him to indulge in repose. But such was not his disposition. In addition to numerous works on the history and legislation of Savoy, he has continued his labors steadily down to the present year. Besides a careful and discriminative eulogy of the distinguished statesman and orator of France, Adolphe Thiers, he lastly carried through the press a learned historical review of the nature and character of the ancient legislative assemblies of Piedmont and Savoy, a thick volume, which had barely reached this country before the news of his decease.

Since the last Report, the Academy has received an accession of twenty-six new members, as follows: twenty Fellows, Charles R. Cross, George Clarke, Amos E. Dolbear, L. Trouvelot, Arthur Searle, in Class I.; Edward Burgess, Thomas P. James, Francis Minot, James J. Putnam, George C. Shattuck, John C. Warren, in Class II.; C. S. Bradley, Phillips Brooks, John Fiske, O. W. Holmes, Jr., C. G. Loring, John Lowell, J. B. Thayer, John W. White, and Justin Winsor, in Class III.; six Foreign Honorary Members, Hofman in place of Poggendorff, Heer in place of Hoffmeister, Leuckart in place of Ehrenburg, Nägeli in place of Alex. Braun, Steenstrup in place of Von Baer, and Plantamour in place of LeVerrier. On the other hand, by removal from the State, or by resignation, the following Fellows have abandoned their membership: F. H. Hedge, E. R. Hoar, John McCrady, Francis Wharton, and J. D. Whitney. Lastly, the following formerly Resident Fellows have been transferred to the list of Associate Fellows: E. B. Elliot, R. Pumpelly, C. S. Peirce, J. Rodgers. The list of the Academy corrected to May 28, 1878, is hereto added. It includes 185 Fellows, 98 Associate Fellows, and 69 Foreign Honorary Members.

LIST

OF THE FELLOWS AND FOREIGN HONORARY MEMBERS.

FELLOWS. — 185.

(Number limited to two hundred.)

Class I. — *Mathematical and Physical Sciences.* — 59.

Section I. — 6.

Mathematics.

Benjamin A. Gould,	Cambridge.
Gustavus Hay,	Boston.
Benjamin Peirce,	Cambridge.
James M. Peirce,	Cambridge.
John D. Runkle,	Boston.
Edwin P. Seaver,	Boston.

Section II. — 10.

Practical Astronomy and Geodesy.

J. Ingersoll Bowditch,	Boston.
Alvan Clark,	Cambridgeport.
George Clark,	Cambridgeport.
Henry Mitchell,	Roxbury.
Robert Treat Paine,	Boston.
E. C. Pickering,	Cambridge.
William A. Rogers,	Cambridge.
Arthur Searle,	Cambridge.
L. Trouvelot,	Cambridge.
Henry L. Whiting,	Boston.

Section III. — 28.

Physics and Chemistry.

John Bacon,	Boston.
A. Graham Bell,	Boston.
John H. Blake,	Boston.
Thos. Edwards Clark,	Williamstown.
W. J. Clark,	Amherst.
Josiah P. Cooke, Jr.,	Cambridge.
James M. Crafts,	Boston.
Charles R. Cross,	Boston.
William P. Dexter,	Roxbury.
Amos E. Dolbear,	Medford.
Charles W. Eliot,	Cambridge.
Moses G. Farmer,	Newport.
Wolcott Gibbs,	Boston.
Augustus A. Hayes,	Brookline.
Henry B. Hill,	Cambridge.
Eben N. Horsford,	Cambridge.
T. Sterry Hunt,	Boston.
Charles L. Jackson,	Cambridge.
Joseph Lovering,	Cambridge.
John M. Merrick,	Boston.
William R. Nichols,	Boston.
John M. Ordway,	Boston.
Edward S. Ritchie,	Boston.
S. P. Sharples,	Cambridge.
Frank H. Storer,	Jamaica Plain.
John Trowbridge,	Cambridge.
Cyrus M. Warren,	Brookline.
Charles H. Wing,	Boston.

Section IV. — 15.

Technology and Engineering.

G. R. Baldwin,	Woburn.
John M. Batchelder,	Cambridge.
C. O. Boutelle,	Washington.
Henry L. Eustis,	Cambridge.
James B. Francis,	Lowell.
John B. Henck,	Boston.
John C. Lee,	Salem.
William R. Lee,	Roxbury.
Hiram F. Mills,	Lawrence.
Alfred P. Rockwell,	Boston.
Stephen P. Ruggles,	Boston.
Charles S. Storrow,	Boston.
William R. Ware,	Boston.
William Watson,	Boston.
Morrill Wyman,	Cambridge.

CLASS II. — *Natural and Physiological Sciences.* — 66.

SECTION I. — 9.

Geology, Mineralogy, and Physics of the Globe.

Thomas T. Bouvé,	Boston.
William T. Brigham,	Boston.
Algernon Coolidge,	Boston.
John L. Hayes,	Cambridge.
Charles T. Jackson,	Boston.
Jules Marcou,	Cambridge.
William B. Rogers,	Boston.
Nathaniel S. Shaler,	Cambridge.
Charles U. Shepard,	Amherst.

SECTION II. — 11.

Botany.

Jacob Bigelow,	Boston.
George B. Emerson,	Boston.
William G. Farlow,	Boston.
George L. Goodale,	Cambridge.
Asa Gray,	Cambridge.
H. H. Hunnewell,	Wellesley.
Thomas P. James,	Cambridge.
John A. Lowell,	Boston.
Chas. J. Sprague,	Boston.
Edward Tuckerman,	Amherst.
Sereno Watson,	Cambridge.

SECTION III. — 26.

Zoölogy and Physiology.

Alex. E. R. Agassiz,	Cambridge.
J. A. Allen,	Cambridge.
Robert Amory,	Brookline.
Nath. E. Atwood,	Provincetown.
James M. Barnard,	Boston.
Henry P. Bowditch,	Boston.
Thomas M. Brewer,	Boston.
Edward Burgess,	Boston.
Samuel Cabot,	Boston.
John Dean,	Waltham.
Silas Durkee,	Boston.
Hermann A. Hagen,	Cambridge.
C. E. Hamlin,	Cambridge.
Alpheus Hyatt,	Cambridge.
Wm. James,	Cambridge.
Samuel Kneeland,	Boston.
Theodore Lyman,	Boston.
Edward S. Morse,	Salem.
Alpheus S. Packard, Jr.,	Salem.
L. F. Pourtales,	Cambridge.
Frederic W. Putnam,	Cambridge.
James J. Putnam,	Boston.
Samuel H. Scudder,	Cambridge.
D. Humphreys Storer,	Boston.
Henry Wheatland,	Salem.
James C. White,	Boston.

SECTION IV. — 20.

Medicine and Surgery.

Samuel L. Abbot,	Boston.
Henry J. Bigelow,	Boston.
Henry I. Bowditch,	Boston.
Benjamin E. Cotting,	Roxbury.
Thomas Dwight,	Boston.
Robert T. Edes,	Roxbury.
Calvin Ellis,	Boston.
Richard M. Hodges,	Boston.
Oliver W. Holmes,	Boston.
R. W. Hooper,	Boston.
John B. S. Jackson,	Boston.
Edward Jarvis,	Dorchester.
Francis Minot,	Boston.
Edward Reynolds,	Boston.
George C. Shattuck,	Boston.
Horatio R. Storer,	Boston.
J. Baxter Upham,	Boston.
Charles E. Ware,	Boston.
John C. Warren,	Boston.
Henry W. Williams,	Boston.

CLASS III. — *Moral and Political Sciences.* — 60.

SECTION I. — 15.

Philosophy and Jurisprudence.

C. S. Bradley,	Cambridge.
Phillips Brooks,	Boston.
Richard H. Dana, Jr.,	Boston.
C. C. Everett,	Cambridge.
John Fiske,	Cambridge.
Horace Gray,	Boston.
L. P. Hickok,	Northampton.
O. W. Holmes, Jr.,	Boston.
Mark Hopkins,	Williamstown.
C. C. Langdell,	Cambridge.
John Lowell,	Boston.
Henry W. Paine,	Cambridge.
Theophilus Parsons,	Cambridge.
J. B. Thayer,	Cambridge.
Benjamin F. Thomas,	Boston.

SECTION II. — 14.

Philology and Archæology.

Ezra Abbot,	Cambridge.
William P. Atkinson,	Boston.
H. G. Denny,	Boston.
Epes S. Dixwell,	Cambridge.
William Everett,	Cambridge.
William W. Goodwin,	Cambridge.
Ephraim W. Gurney,	Cambridge.
Bennett H. Nash,	Boston.
Chandler Robbins,	Boston.
John L. Sibley,	Cambridge.
E. A. Sophocles,	Cambridge.
John W. White,	Cambridge.
Justin Winsor,	Cambridge.
Edward J. Young,	Cambridge.

SECTION III. — 16.

Political Economy and History.

Chas. F. Adams, Jr.,	Quincy.
Henry Adams,	Boston.
Erastus B. Bigelow,	Boston.
Caleb Cushing,	Newburyport.
Charles Deane,	Cambridge.
Charles F. Dunbar,	Cambridge.
Samuel Eliot,	Boston.
George E. Ellis,	Boston.
E. L. Godkin,	Cambridge.
William Gray,	Boston.
Edward Everett Hale,	Boston.
Francis Parkman,	Brookline.
A. P. Peabody,	Cambridge.
Nathaniel Thayer,	Boston.
Henry W. Torrey,	Cambridge.
Robert C. Winthrop,	Boston.

SECTION IV. — 15.

Literature and the Fine Arts.

Charles F. Adams,	Boston.
William T. Andrews,	Boston.
George S. Boutwell,	Groton.
J. Elliot Cabot,	Brookline.
Francis J. Child,	Cambridge.
Ralph Waldo Emerson,	Concord.
John C. Gray,	Cambridge.
Henry W. Longfellow,	Cambridge.
Charles G. Loring,	Boston.
James Russell Lowell,	Cambridge.
Charles Eliot Norton,	Cambridge.
John K. Paine,	Cambridge.
Thomas W. Parsons,	Wayland.
Charles C. Perkins,	Boston.
John G. Whittier,	Amesbury.

ASSOCIATE FELLOWS. — 98.

(Number limited to one hundred.)

CLASS I. — *Mathematical and Physical Sciences.* — 37.

SECTION I. — 8.

Mathematics.

Charles Avery,	Clinton, N.Y.
E. B. Elliott,	Washington, D.C.
William Ferrel,	Washington, D.C.
Thomas Hill,	Portland, Me.
Simon Newcomb,	Washington, D.C.
H. A. Newton,	New Haven, Conn.
James E. Oliver,	Ithaca, N.Y.
T. H. Safford,	Williamstown, Mass.

SECTION II. — 11.

Practical Astronomy and Geodesy.

S. Alexander,	Princeton, N.J.
W. H. C. Bartlett,	West Point, N.Y.
J. H. C. Coffin,	Washington, D.C.
Wm. H. Emory,	Washington, D.C.
J. E. Hilgard,	Washington, D.C.
George W. Hill,	Nyack, N.Y.
Elias Loomis,	New Haven, Conn.
Maria Mitchell,	Poughkeepsie, N.Y.
C. H. F. Peters,	Clinton, N.Y.
George M. Searle,	New York.
Chas. A. Young,	Princeton, N.J.

SECTION III. — 11.

Physics and Chemistry.

F. A. P. Barnard,	New York.
John W. Draper,	New York.
S. W. Johnson,	New Haven, Conn.
John Le Conte,	San Francisco, Cal.
A. M. Mayer,	Hoboken, N.J.
W. A. Norton,	New Haven, Conn.
Ogden N. Rood,	New York.
H. A. Rowland,	Baltimore.
L. M. Rutherfurd,	New York.
Benj. Silliman,	New Haven, Conn.
J. L. Smith,	Louisville, Ky.

SECTION IV. — 7.

Technology and Engineering.

Henry L. Abbot,	New York.
R. Delafield,	Washington, D.C.
A. A. Humphreys,	Washington, D.C.
John Rodgers,	Washington, D.C.
Wm. Sellers,	Philadelphia.
George Talcott,	Albany, N.Y.
W. P. Trowbridge,	New Haven, Conn.

CLASS II. — *Natural and Physiological Sciences.* — 29.

SECTION I. — 14.

Geology, Mineralogy, and Physics of the Globe.

George J. Brush,	New Haven, Conn.
James D. Dana,	New Haven, Conn.
J. W. Dawson,	Montreal, Canada.
Edward Desor,	Neufchâtel, Switz.
J. C. Fremont,	New York.
F. A. Genth,	Philadelphia.
Arnold Guyot,	Princeton, N.J.
James Hall,	Albany, N.Y.
F. S. Holmes,	Charleston, S.C.
Joseph Leconte,	San Francisco.
J. Peter Lesley,	Philadelphia.
R. Pumpelly,	Owego, N.Y.
Wm. T. Roepper,	Bethlehem, Pa.
Geo. C. Swallow,	Columbia, Mo.

SECTION II.—4.

Botany.

A. W. Chapman,	Apalachicola, Fla.
G. Engelmann,	St. Louis, Mo.
Leo Lesquereux,	Columbus, Ohio.
S. T. Olney,	Providence, R.I.

SECTION III.—8.

Zoölogy and Physiology.

S. F. Baird,	Washington, D.C.
C. E. Brown-Séquard,	London.
J. C. Dalton,	New York.
J. L. LeConte,	Philadelphia.
Joseph Leidy,	Philadelphia.
O. C. Marsh,	New Haven, Conn.
S. Weir Mitchell,	Philadelphia.
St. Julien Ravenel,	Charleston, S.C.

SECTION IV.—3.

Medicine and Surgery.

W. A. Hammond,	New York.
Isaac Hays,	Philadelphia.
George B. Wood,	Philadelphia.

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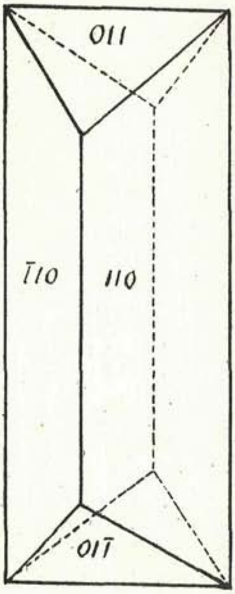
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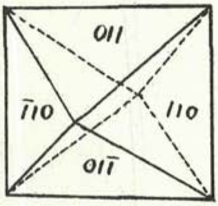
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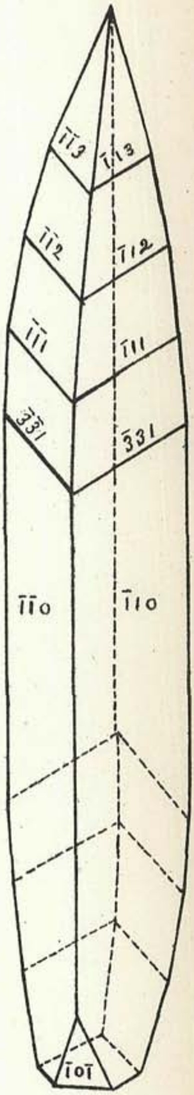
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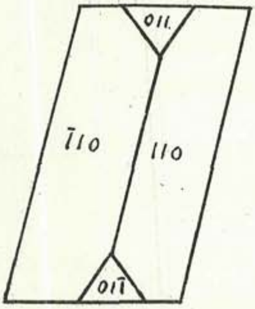
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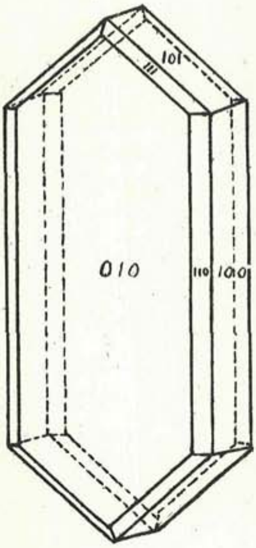
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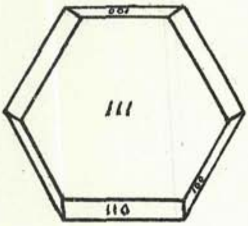
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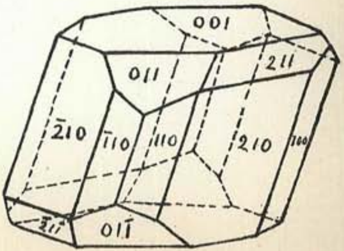
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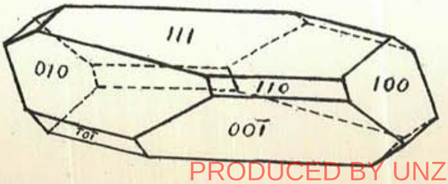
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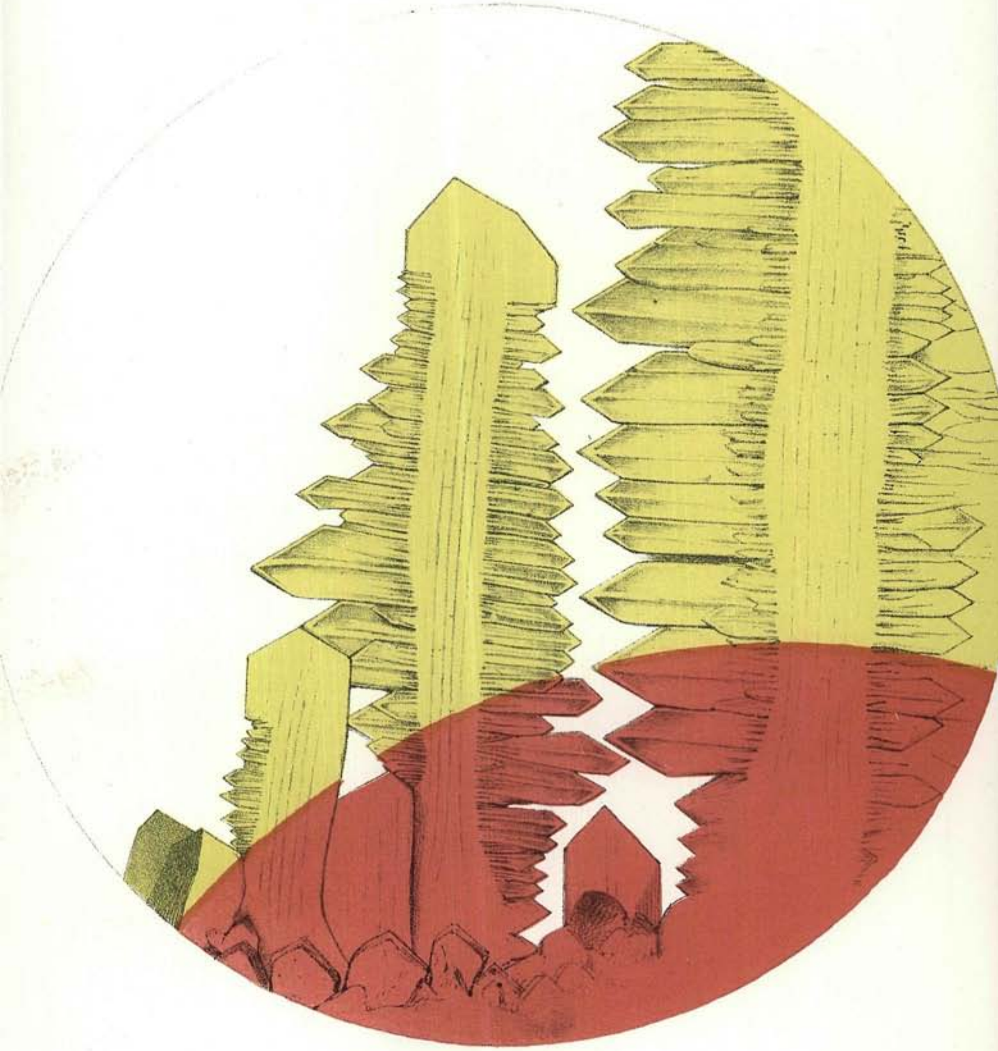

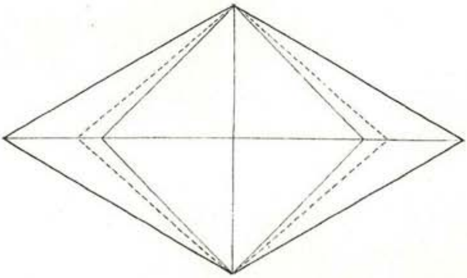

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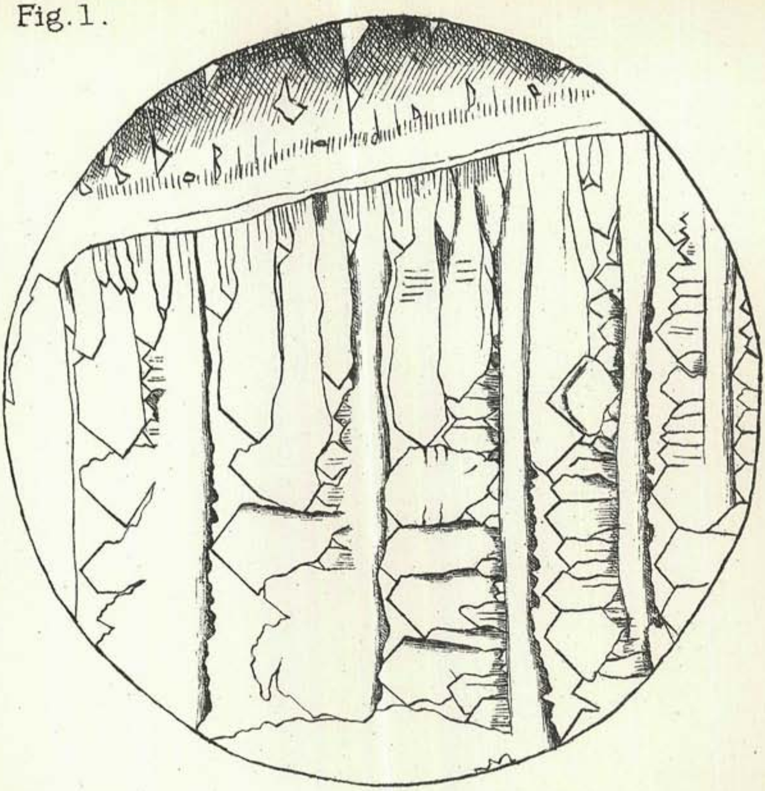

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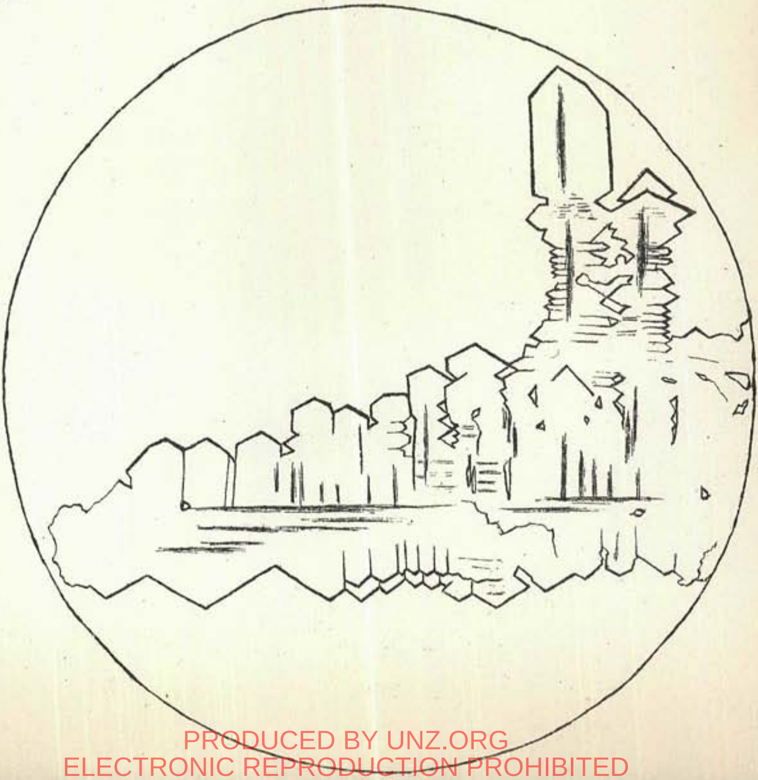

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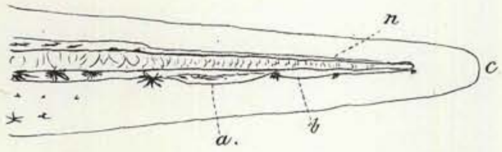

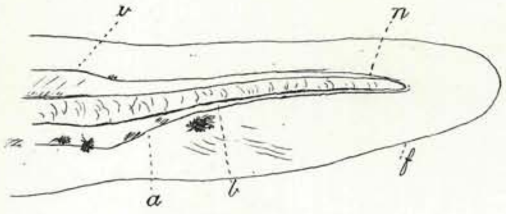

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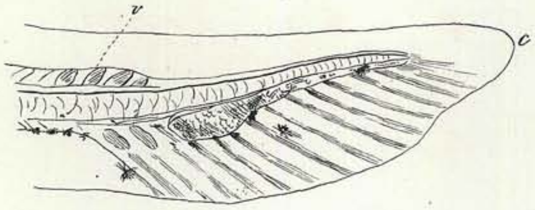

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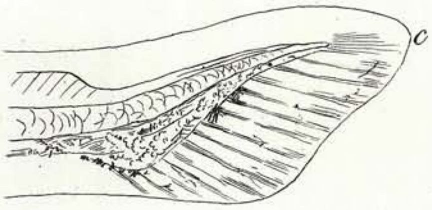

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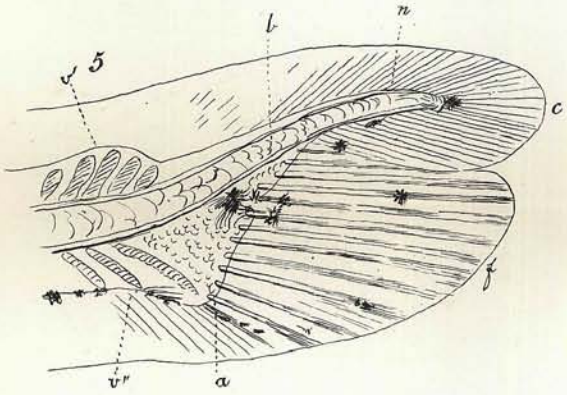

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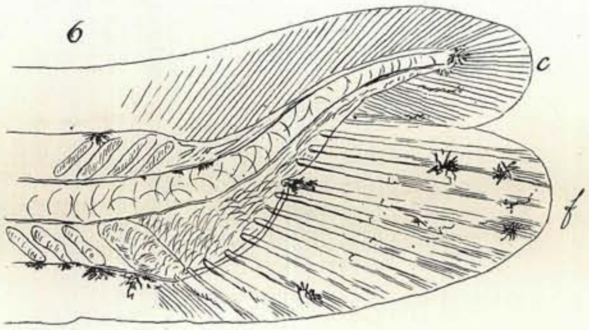

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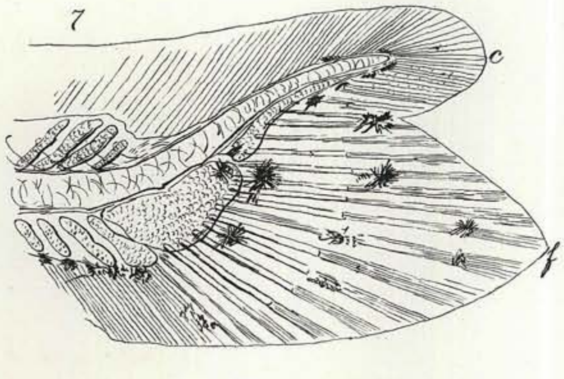

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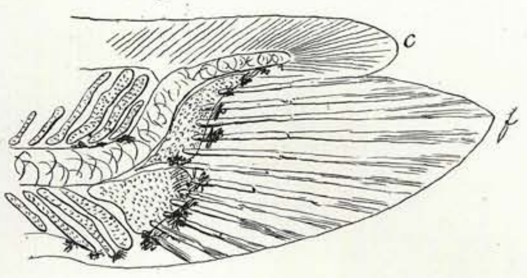

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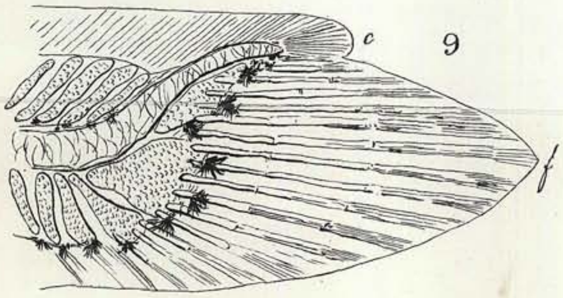

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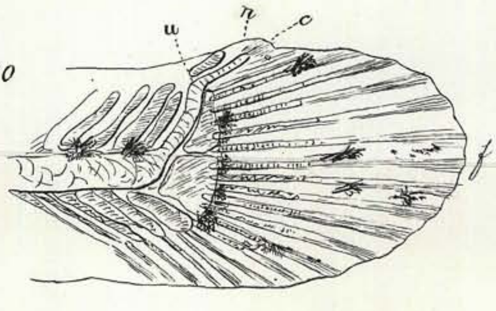

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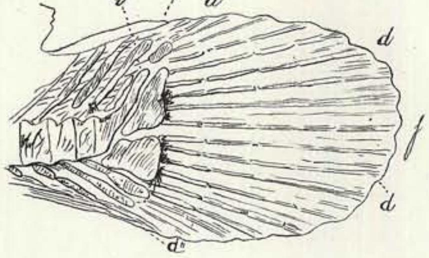

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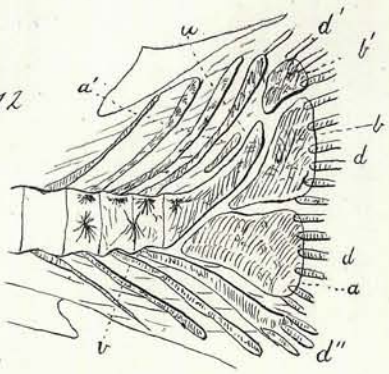
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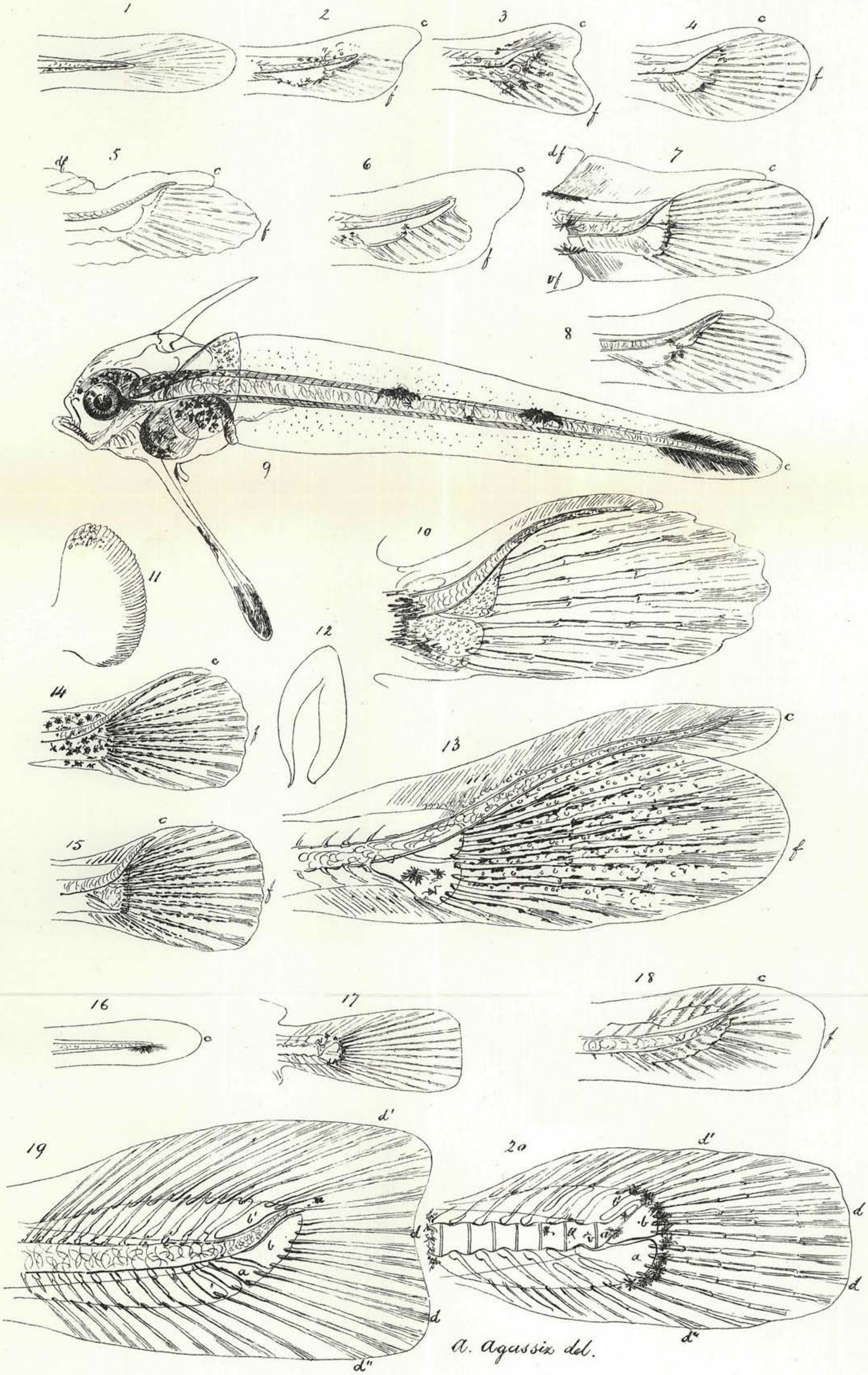
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A. Agassiz del.

FELDSPAR DETERMINATIONS.

	Measurements of the entering angle on the basal cleavage plane. (*Goniometer.*)			Angle between the principal sections of adjacent hemitropic bands in sections cut parallel to O. (Angle between maximum extinctions.) (*Des Cloiseaux's Method.*)		Angle between the principal sections of adjacent hemitropic bands in sections cut at random in the zone *O : i ĭ*. (*Des Cloiseaux's Method.*)		Specific gravity of the Rock.	Probable feldspar.
Lower zone of bed 22, Eagle R. Section						34° 34° 36° 51° 60° 62°	Labr.	. . .	Labradorite.
Coarse-grained diabase, north of Delaware Mine						22° 30° 41° 47° 50° 60° 71°	Anorth.	. . .	Anorthite.
Coarse-grained diabase, north of Delaware Mine, equiv. of bed 94						22° 23° 31° 33° 33° 33° 36°	Olig. or Lab.	. . .	Oligoclase.
Lower zone of bed 94, Eagle R. Section	Entering angle on basal cleavage	172° 50′ — 173° 12′	Alb. Olig.	Extinction Angle ± 3°	Olig. And.	29° 30° 31° 31° 34° 37° 38° 41°	Olig. or Lab	2.94	Oligoclase.
	Angle measured *O* | *i ĭ*	93° 24′	Alb Olig.					. . .	
	Same angle measured under microscope	93° 29′						. . .	
	Another crystal under microscope *O* | *i ĭ*	90° ± 10′	Orthoclase.						Orthoclase.
Coarse-grained diabase, west of Quincy Mine						43° 46° 48° 49° 55°	Lab. or Ano.	. . .	Lab. or Anorth.
Coarse-grained diabase, mouth of adit at the Dacotah Mine				Ext'n. angle, two-thirds is on one side in each instance 23° 43°	Anorth.	59° 62° 63° 66° 66° 78°	Anorth.	. . .	Anorthite.
Bed 96, Eagle R. Section; diabase. Feldspar prehnitized	Entering angle	172° 09′ — 172° 49′	Alb Olig. or Labr.	Angle between line of ext'n. and edge *O* | *i ĭ* 6° & 0° In other sets of bands 6° and 3°	Lab. Alb.	6° 7° 11° 13° 31° 45° 58° 20° 20° 27°	Lab. Anor.	2.90	Labradorite.
Bed 98, Eagle R. Section: diabase. Feldspar prehnitized				Extinction angle all in *one set* of bands 6°	Lab. Alb.	16° 20° 41°	Lab.	. . .	Labradorite.
Bed 107 A, Eagle R. Section (light type), diabase. Feldspar prehnitized		171° 30′ — 172° 21′	Anor. Olig			10° 11° 24° 29° 39° 56° 57° 60°	Lab. Anor.	Rock 3.03 Feldsp. 2.73 Aug. 3.39	Labradorite.
So-called Syenite. Granite St. Quarry, Somerville, Mass.: locality known for its prehnite. Feldspar much altered to prehnite				Extinction angle 5° 54′ to 8°. Sect. oblique. Angle in one set 2 to 3 times as large as in other	Alb. Olig.	24° 26° 31° 32°	Alb. Olig.	. . .	Alb. or Olig.
Lower zone of bed 87, Eagle R. Section. Diabase		171° 26′ — 172° 33′	Olig. Alb.			22° 23° 34° 36°	Alb Olig.	2.6–2.73 too low for Labradorite.	Alb. or Olig SiO^2 46.32 %.†
Middle ,, ,, ,, ,, ,, ,,						26° 29° 34° 35° 36° 36°	Alb. or Olig.	2.73–2.75	Alb. or Olig.
Bed 45, ,, ,, ,,						21° 21° 24° 28° 31° 34°	Alb. or Olig.	2.91	Alb. or Olig.
Lower zone of bed 66, ,, ,, ,,						5° 15° 15° 28° 31° 35°	Alb. or Olig.	2.93	Alb. or Olig.
Fine-grained, hard diabase, Southside location						61° 66° 66° 68° 70°	Anorth.	. . .	Anorthite.
Bed 106, Feldspar segregation. Diabase		170° 54′ — 171° 06′	Anorth.	13° 30′ — 16°	Labr.	11° 12° 18° 29° 31° 62° 64°	Lab. An.	. . .	Labradorite.
Bed 107 B., (dark type), Eagle R. Section. Diabase						16° 20° 30° 49° 60° 70° 75°	Anorth.	. . .	Anorthite.
Lower zone of bed 108, Eagle R. Section. "The Greenstone." Melaphyre		171° 58′ — 172° 25′	Anor. Olig.			43° 43° 48° 49° 50° 55° 58° 60° 60° 63° 69° 70°	Anorth.	2.95	SiO^2 50.20 %.‡ Anorthite.
Bed 90, Eagle River Section. Melaphyre						38° 49° 57° 60° 62° 66°	An. Lab.	2.89	Anorthite.
Bed 64, (lower zone), Eagle R. Section. Melaphyre						47° 64° 67° 73° 74°	Anorth.	2 65–2 70	Anorthite.
Bed 65, under the "ash bed," Eagle R. Section. Diabase or diab. porphyry						38° 55° 60°	Lab. or An.	2.98	Lab. or Anorth.
8th Bed, west of Pewabic, west conglomerate, St. Mary's location. Diabase or diab. porphyry						47° 48° 49°	Lab. or An.	. . .	Lab. or Anorth.
Bed 95, Eagle River Section. Diabase				10° — 17° and 13°	Labr.	39° 50° 51° 51°	Lab. An.	. . .	Labradorite.
Light-colored diabase, "800 ft. E. of Hungarian Lode"						*25° 27° 34° 36°	Alb. Olig.	. . .	Alb. or Olig.
Bed 61 (lower zone), Eagle R. Section. Analcitic.						31° 43° 64° 66°	Lab. An.	. . .	Lab. or Anorth.
Diabase near Houghton ("y. bowld.") Feldspar, changed to prehnite and chlorite						10° 19° 19° 20° 23° 23° 27° 28° 28° 29°	Alb. or Ol.	. . .	Alb. or Olig.
Coarse-grained rock from Sec. 5, T. 47, R. 46, Mich.		171° 41′ — 171° 46′	Anorth	7°		10° 17° 27° 42° 43° 48° 62° 65°	Lab. An.	. . .	Labradorite.
Pseudo-amygdaloid, 50 ft. west of Houghton conglomerate. Mesnard location						44° 61° 64° 77°		. . .	Anorthite.

* Very unequal. † In the rock. ‡ In the rock. Whitney Geol. L. Superior.